# Steiner Ratio for Rectilinear Obstacle-avoiding Steiner Trees

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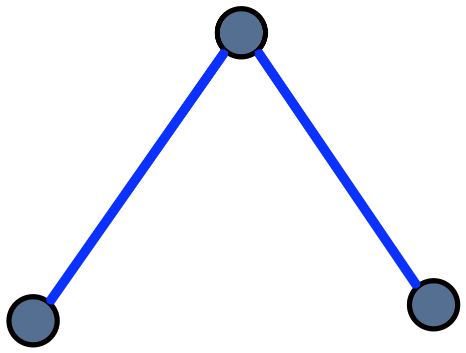
# Steiner Tree Problem

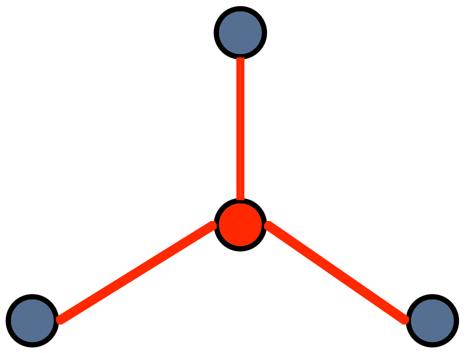
Given : A set P of points in plane→Terminals

Find : A shortest network interconnecting P

⟶ Steiner Minimum Tree (SMT)



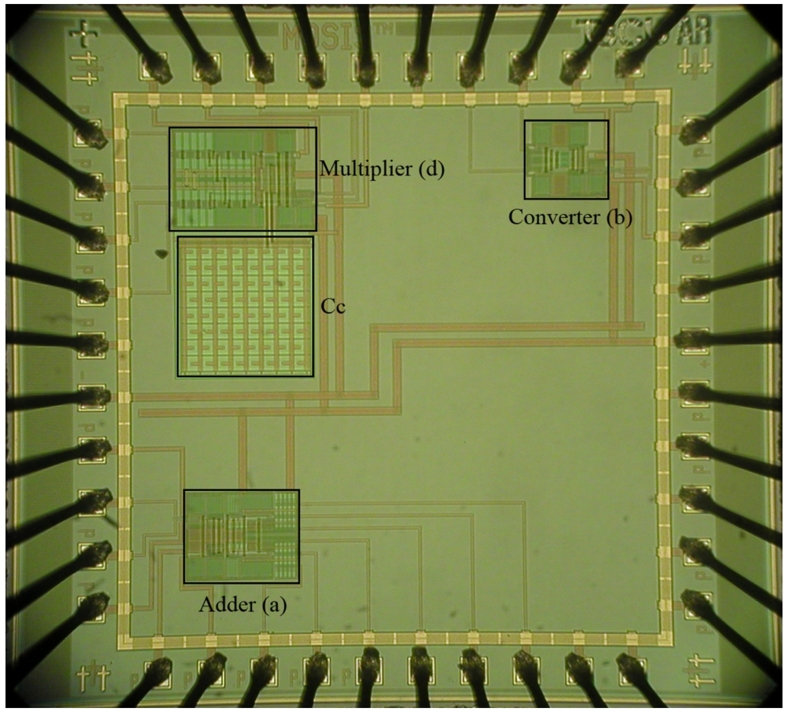
Minimum Spanning Tree



Steiner Minimum Tree

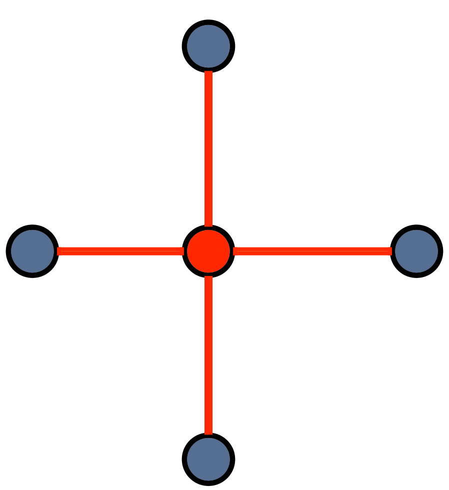
# Motivation

- VLSI design
- Network routing
- Wireless communication
- Computational biology



# Rectilinear Steiner Tree

- Distances measured in $L_1$ metric
- Uses only horizontal and vertical segments
- Motivation : VLSI design



# Obstacle-Avoiding Steiner Trees

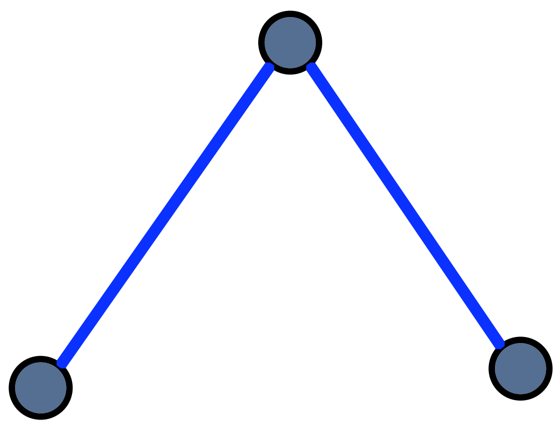
- A set of polygonal/rectangular obstacles.
- Avoid running through the interior of the obstacles.

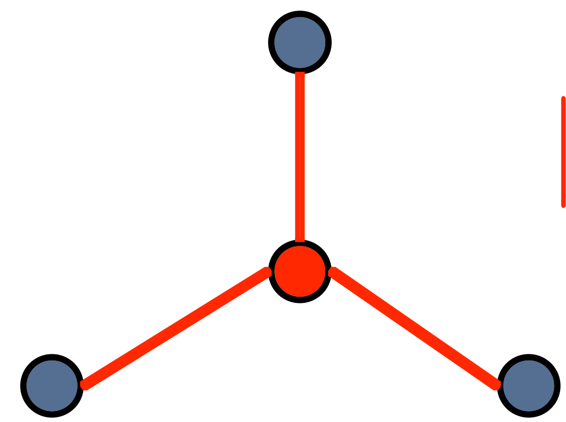
# Steiner Ratio

- Steiner tree problem is NP-hard $\longrightarrow$ Approximation
- No Steiner points $\longrightarrow$ Minimum spanning tree
- Performance Ratio ?

# MST & SMT for 3 Points

$|M| = 2$





$|S| = \sqrt{3}$

$$\frac{|S|}{|M|} = \frac{\sqrt{3}}{2} \simeq 0.866$$

# Steiner Ratio

$$\rho = \inf_P \frac{|S(P)|}{|M(P)|}$$

# Gillbert & Pollak Conjecture (1968)

For any terminal set $P$ in Euclidean plane:

$$\rho(P) = \frac{|S(P)|}{|M(P)|} \leq \frac{\sqrt{3}}{2} \simeq 0.866$$

# Small Cases

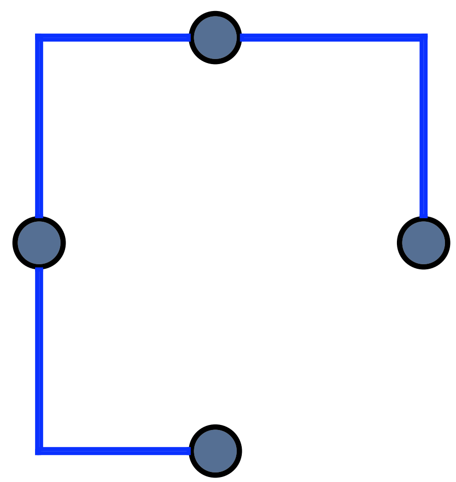
- n =3 [Gilbert, Pollak 1968]
- n = 4 [Pollak 1978]
- n = 5 [Du, Hwang, Yao 1985]
- n = 6 [Rubinston, Thomas 1991]

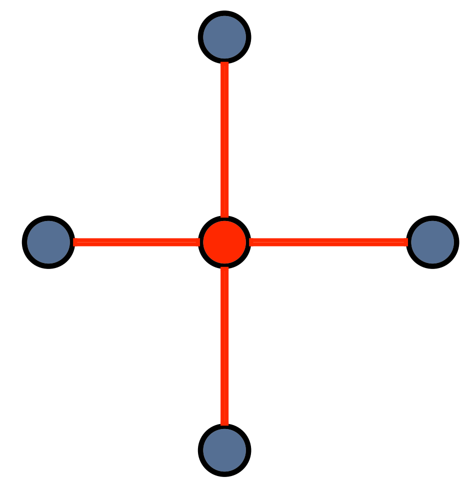
# Lower Bounds

- 0.57 [Graham, Hwang 1976]
- 0.74 [Chung, Hwang 1978]
- 0.8 [Du, Hwang 1983]
- 0.824 [Chung, Graham 1985]
- Finally proved by Du and Hwang in 1991

# RMST & RSMT for 4

$|M| = 6$





$|S| = 4$

$$\frac{|S|}{|M|} = \frac{2}{3}$$

# Rectilinear Steiner Ratio

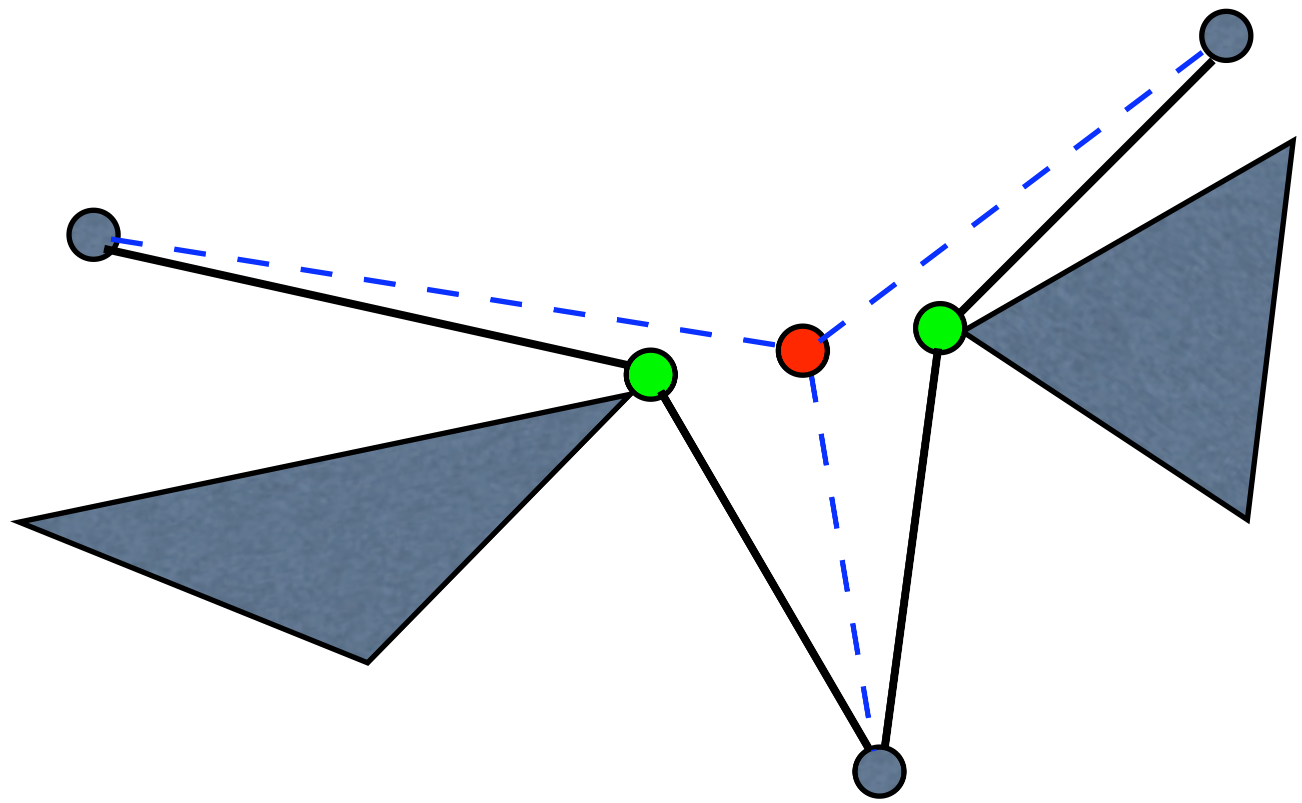
Theorem :

For any terminal set $P$ in rectilinear plane:

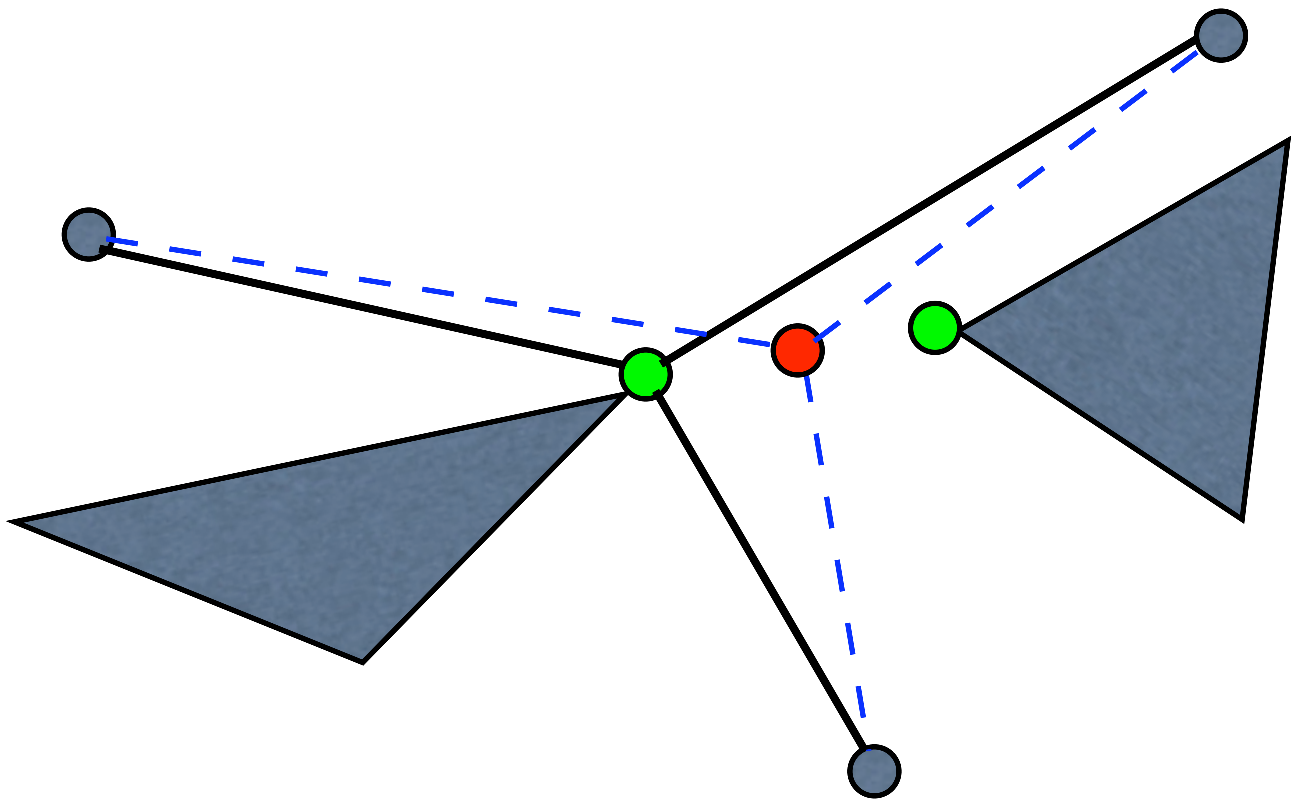
$$\rho(P) = \frac{|S(P)|}{|M(P)|} \leq \frac{2}{3}$$

[Hwang 1976]

# Obstacles



# Obstacles



Allow Steiner points only at obstacle corners.

- Anchored Steiner Tree : All Steiner points are at obstacle corners.

- No obstacles: Minimum spanning tree

# Obstacle-Avoiding Rectilinear Steiner Ratio

Theorem:

$$\rho = \max_P \frac{|S(P)|}{|AS(P)|} = \frac{2}{3}$$

$S(P)$ : Minimum Steiner tree

$AS(P)$ : Minimum anchored Steiner tree

# Hwang's Proof

For any terminal set $P$ in rectilinear plane:

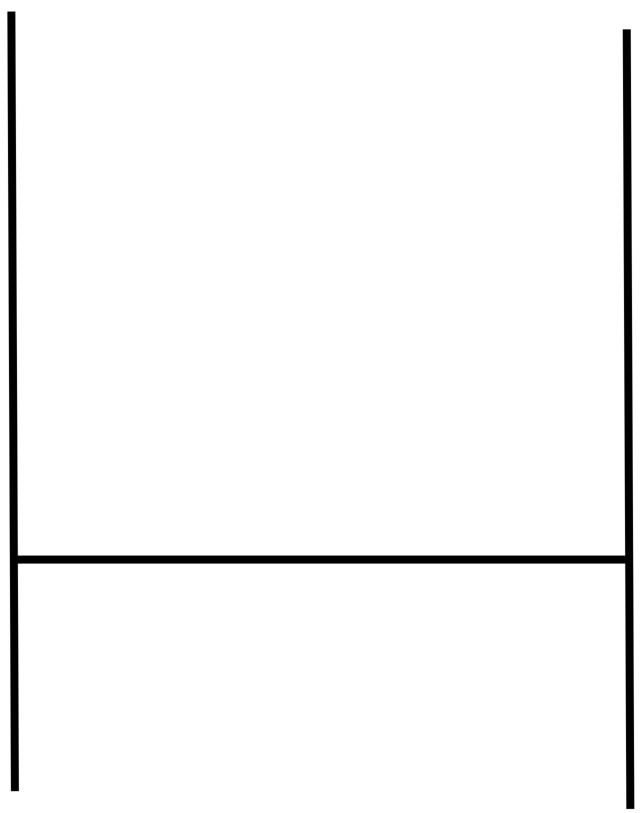
$$\frac{|S(P)|}{|M(P)|} \leq \frac{2}{3}$$

[Hwang 1976]

Proof Stages:

- Canonical trees
- Spanning tree construction

# Shift & Flip



Shift

Flip

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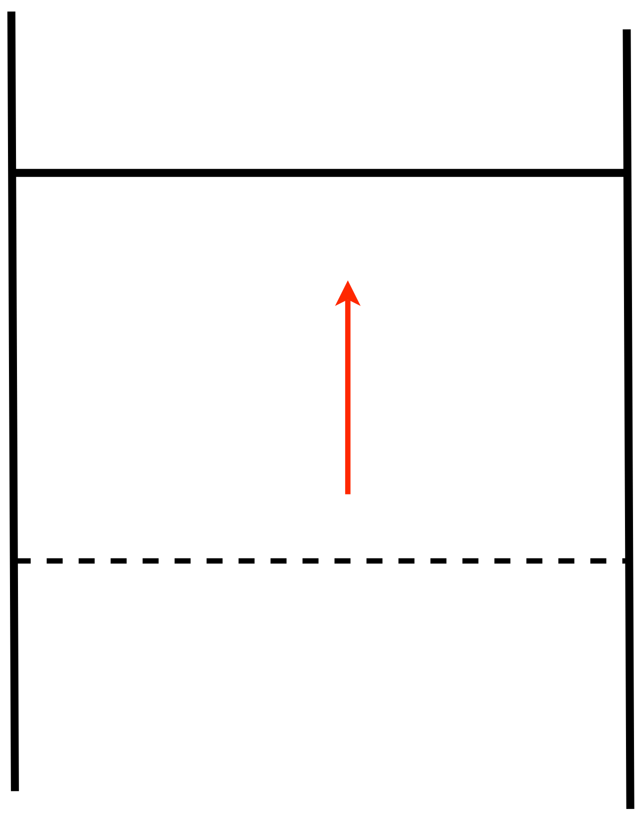

Shift

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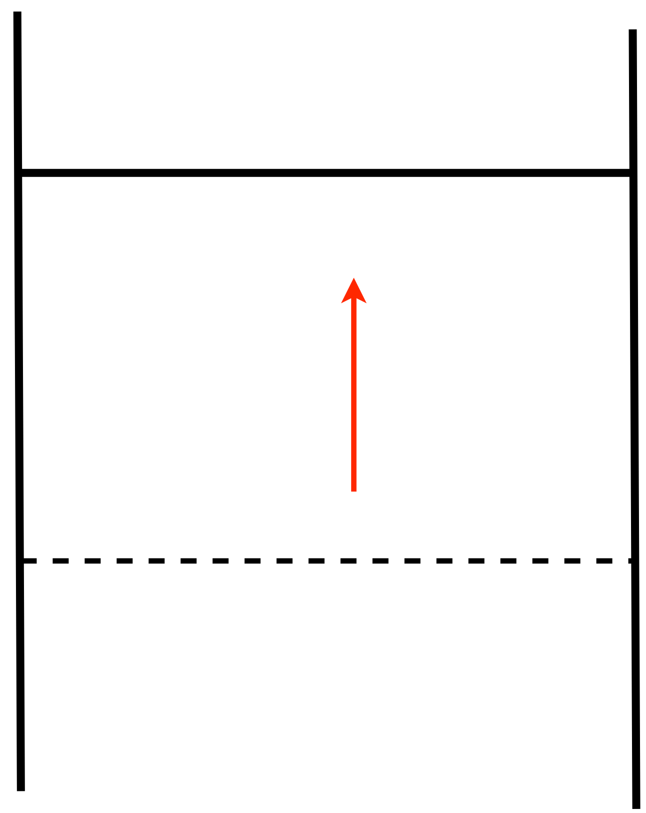
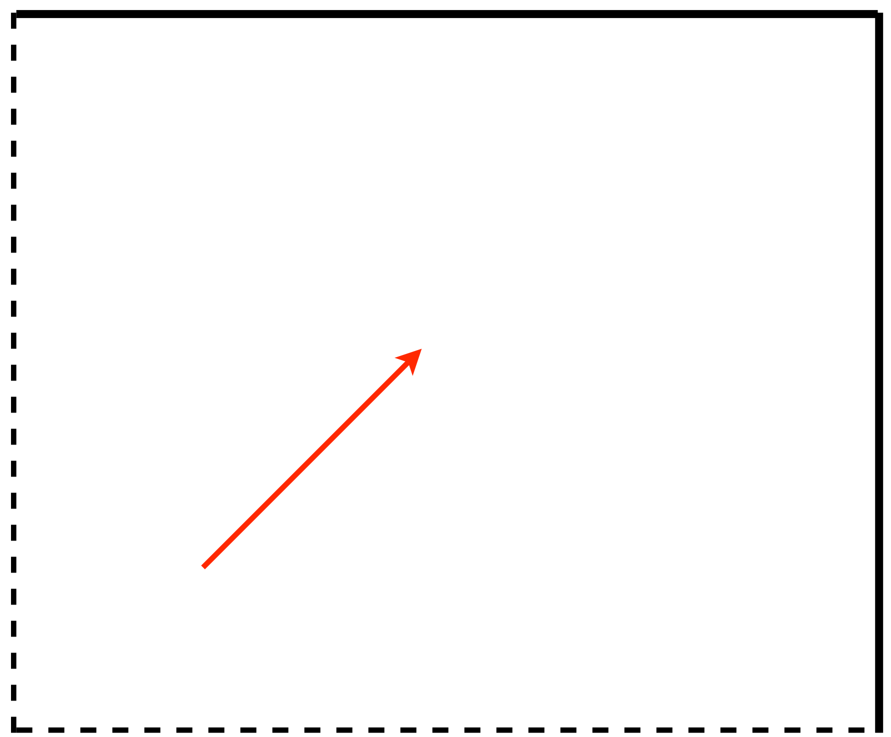


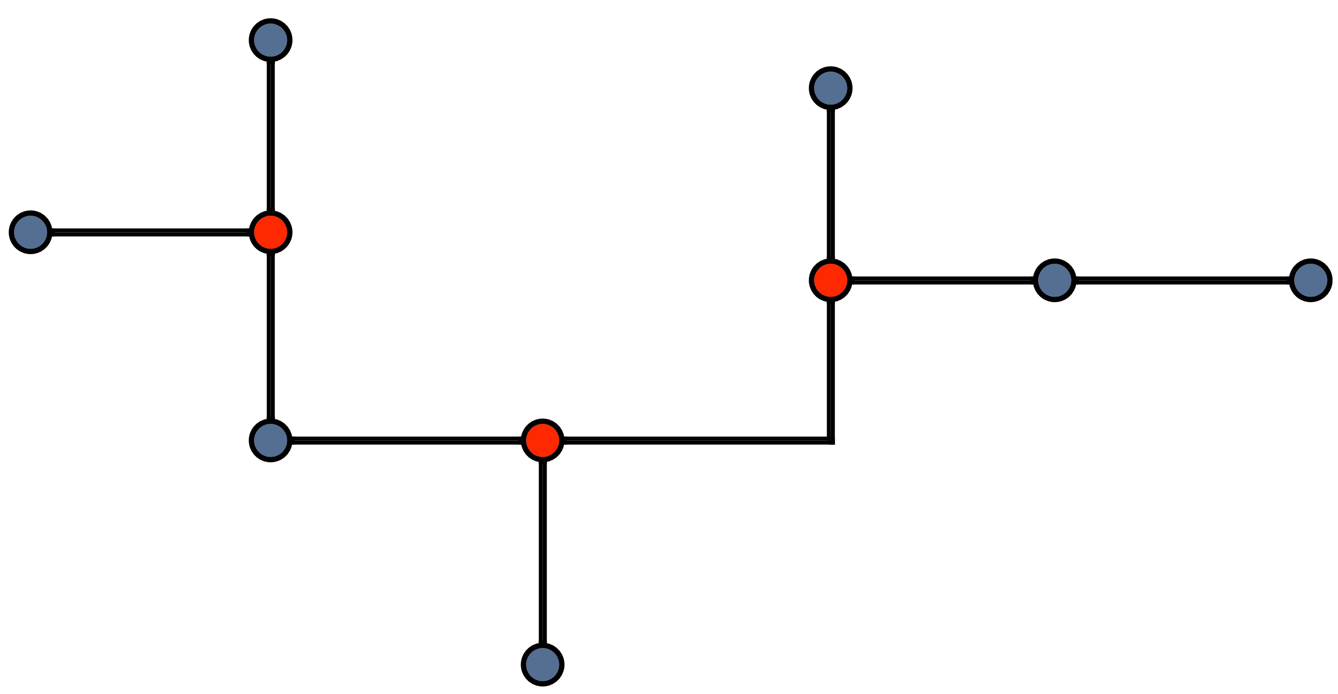
Shift



Flip

# Full Components

Full Steiner Tree : all terminals are leaves.



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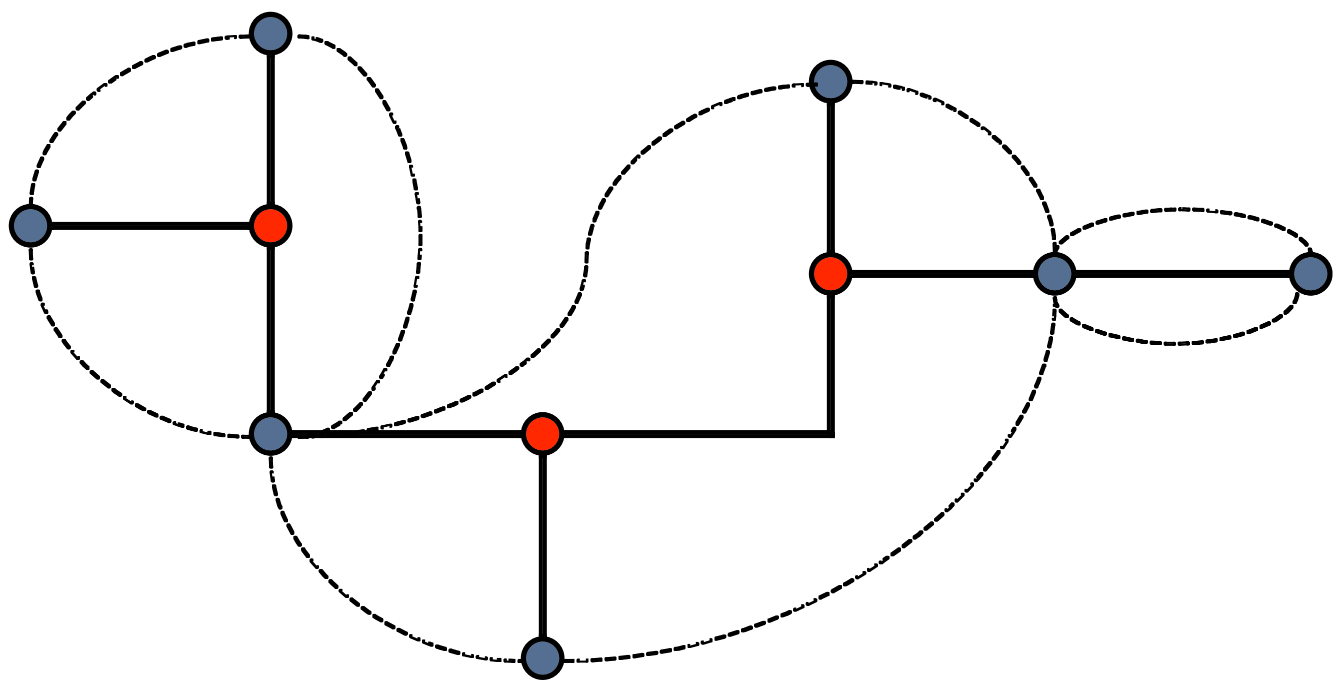
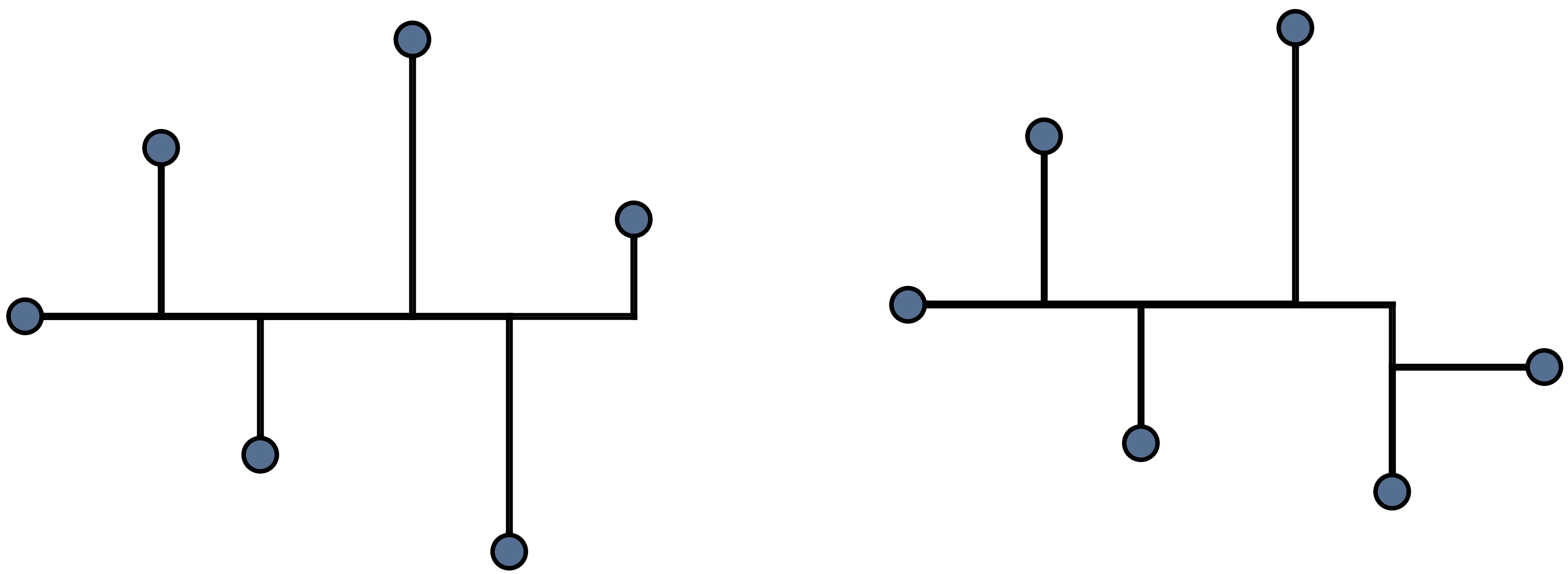


# Canonical Trees

A full component in an SMT with maximum number of full components can be transformed to one of these forms:



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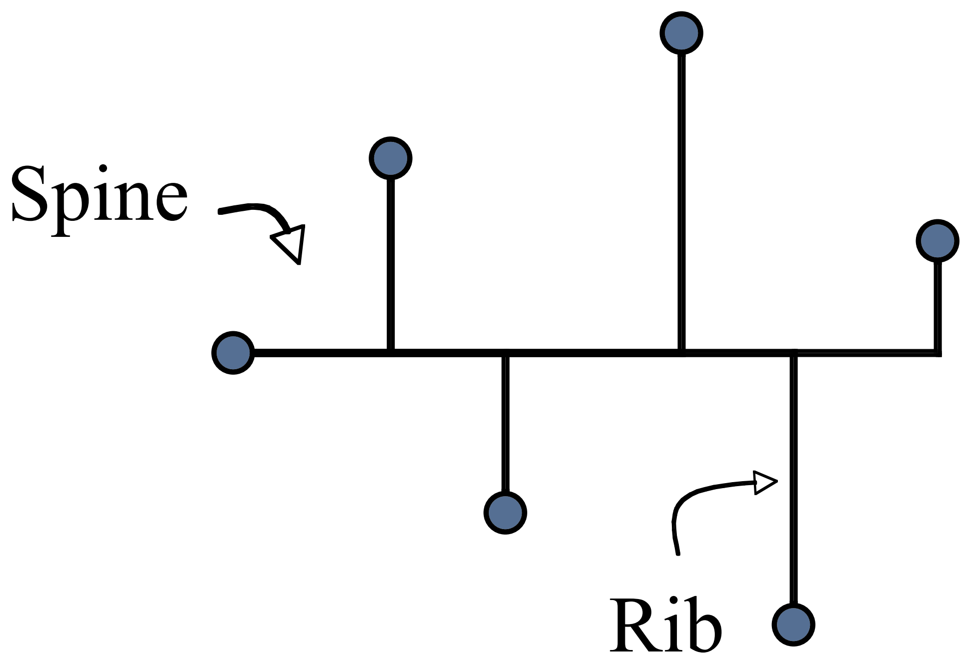
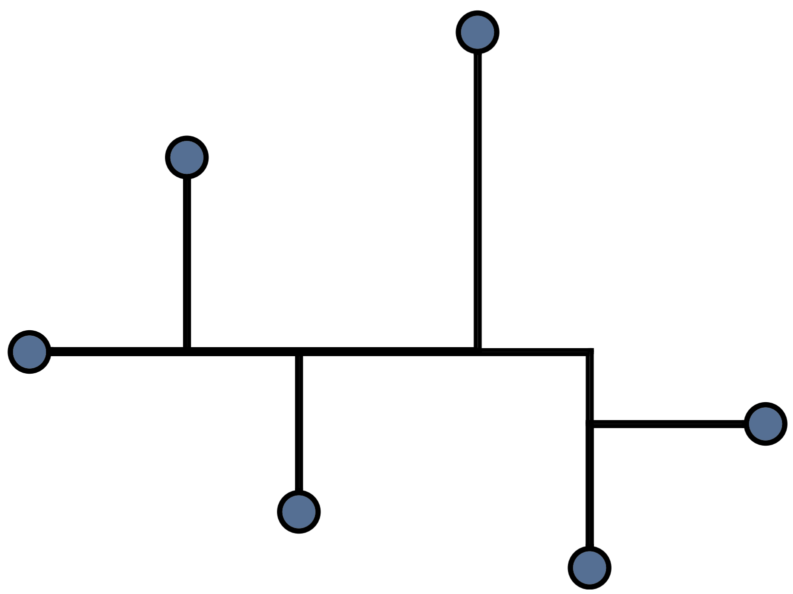
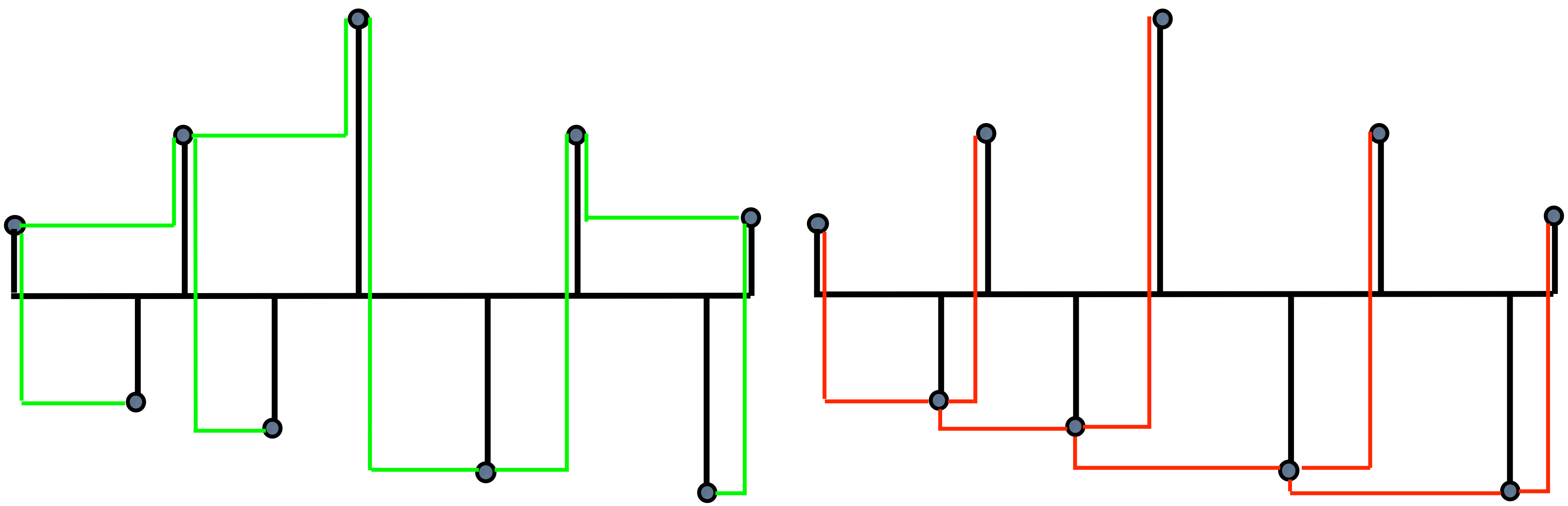


# Spanning Trees [Berman et al.]



$$|T_{red}| + |T_{green}| \leq 3|S|$$

$$|M| \leq \frac{|T_{red}| + |T_{green}|}{2} \leq \frac{3}{2}|S|$$

## Theorem:

For any terminal set $P$ and obstacle set $O$ in rectilinear plane:

$$|AS(P)| \leq \frac{3}{2}|S(P)|$$

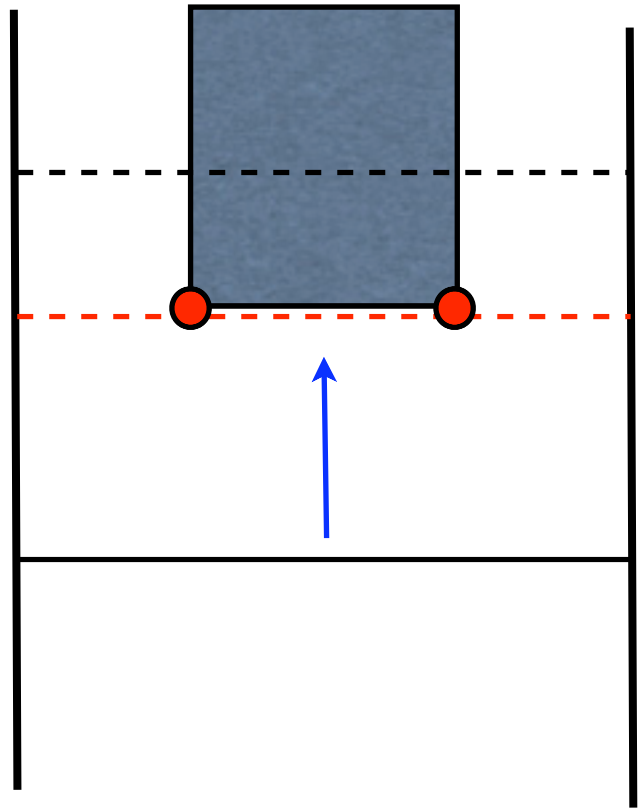
Proof Stages:

- Obstacle-avoiding canonical trees
- Anchored Steiner tree construction

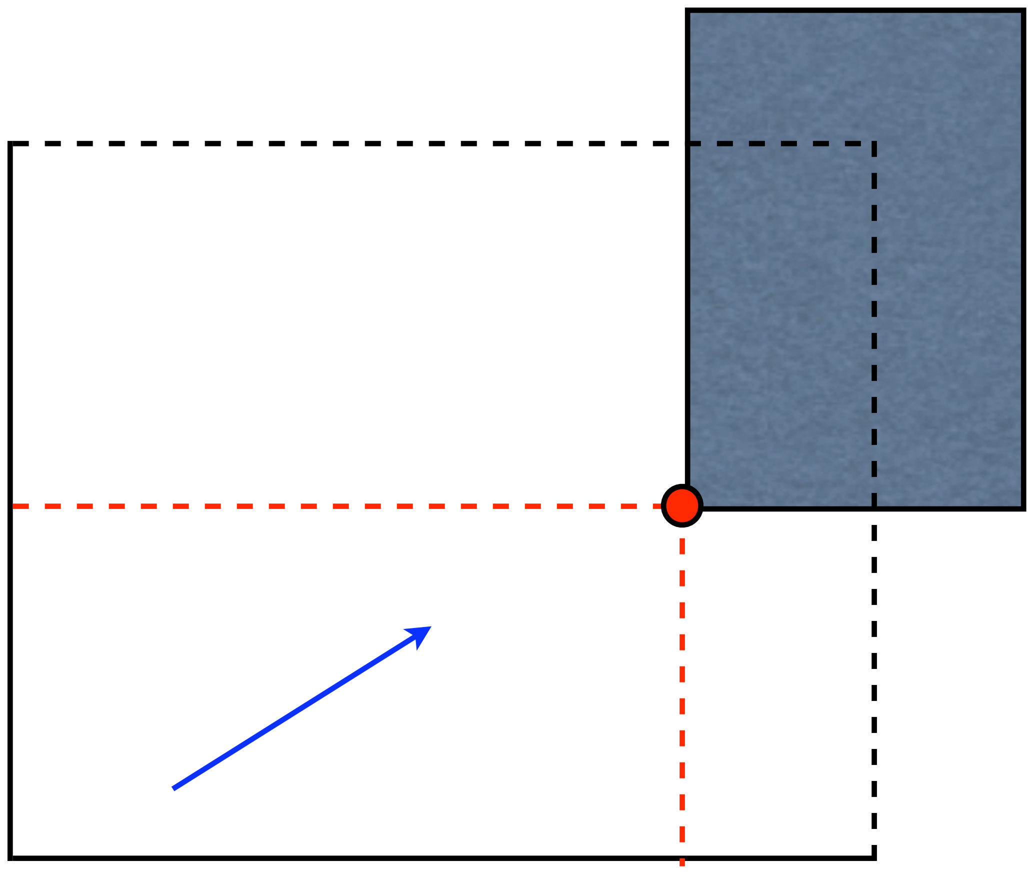
# Obstacle-Avoiding Canonical Trees

- Perform shifts & flips to bump into as many obstacles as possible.
- Split the Steiner tree at non-leaf terminals and obstacle corners.
- The resulting tree has maximum number of full components.

Shifts and flips can ignore the obstacles:

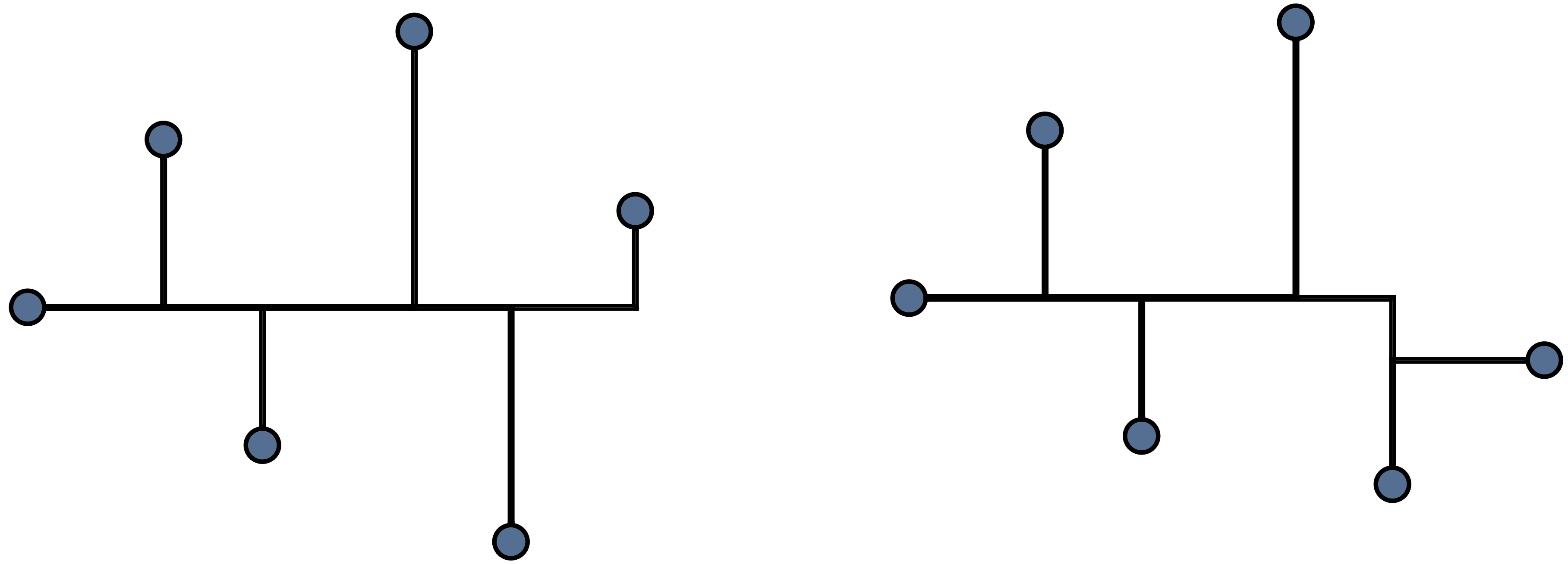


Shift

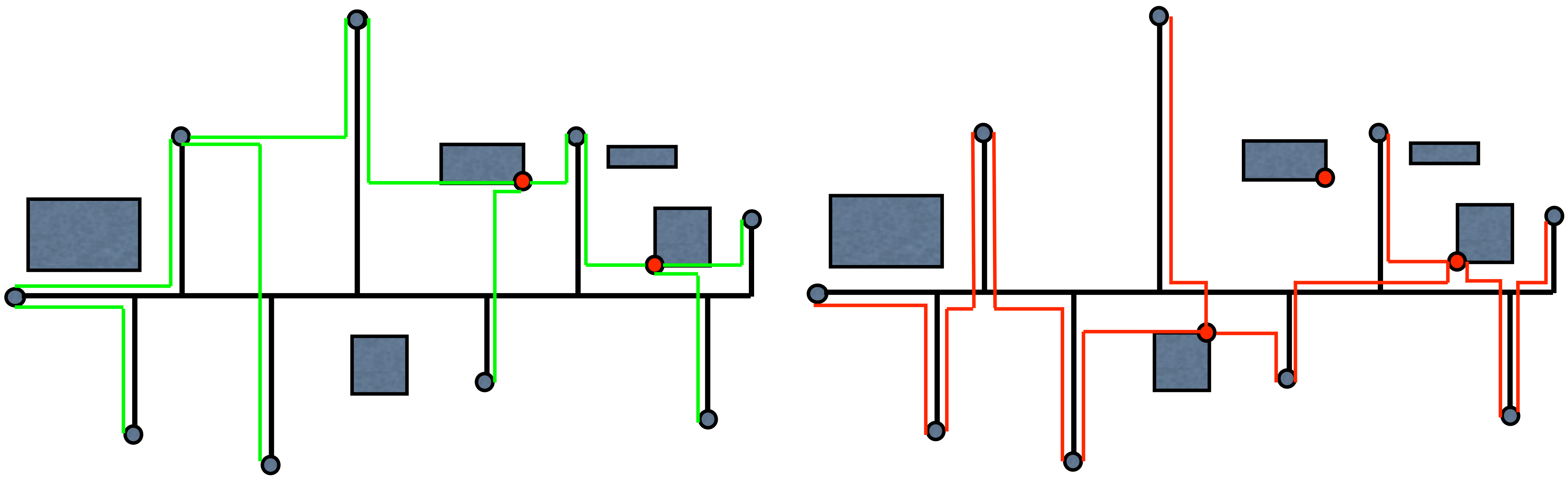


Flip

# Obstacle-Avoiding Canonical Trees



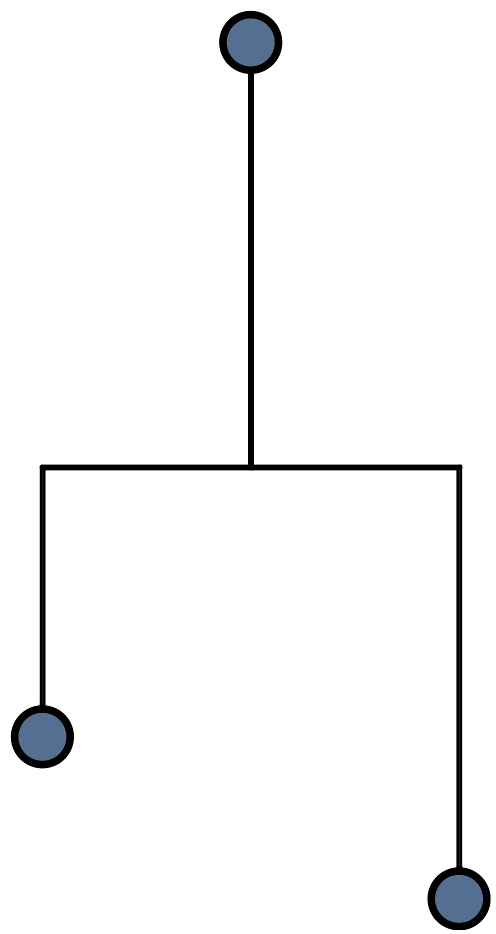
# Anchored Steiner Tree



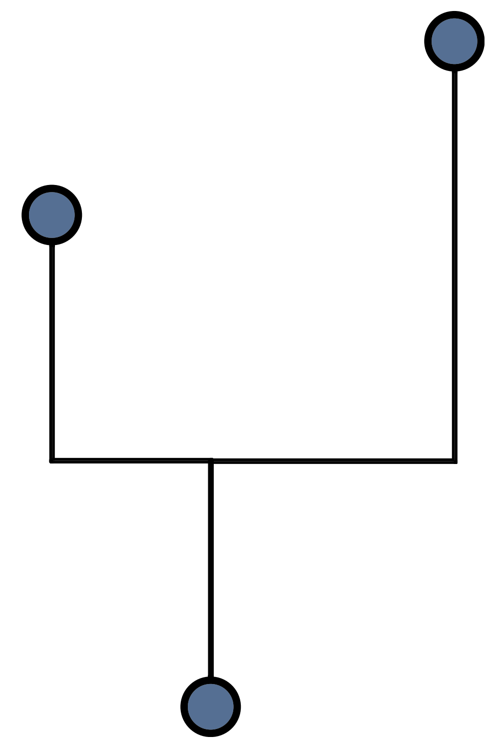
$$|T_{red}| + |T_{green}| \leq 3|S|$$

$$|AS| \leq \frac{|T_{red}| + |T_{green}|}{2} \leq \frac{3}{2}|S|$$

# Pockets

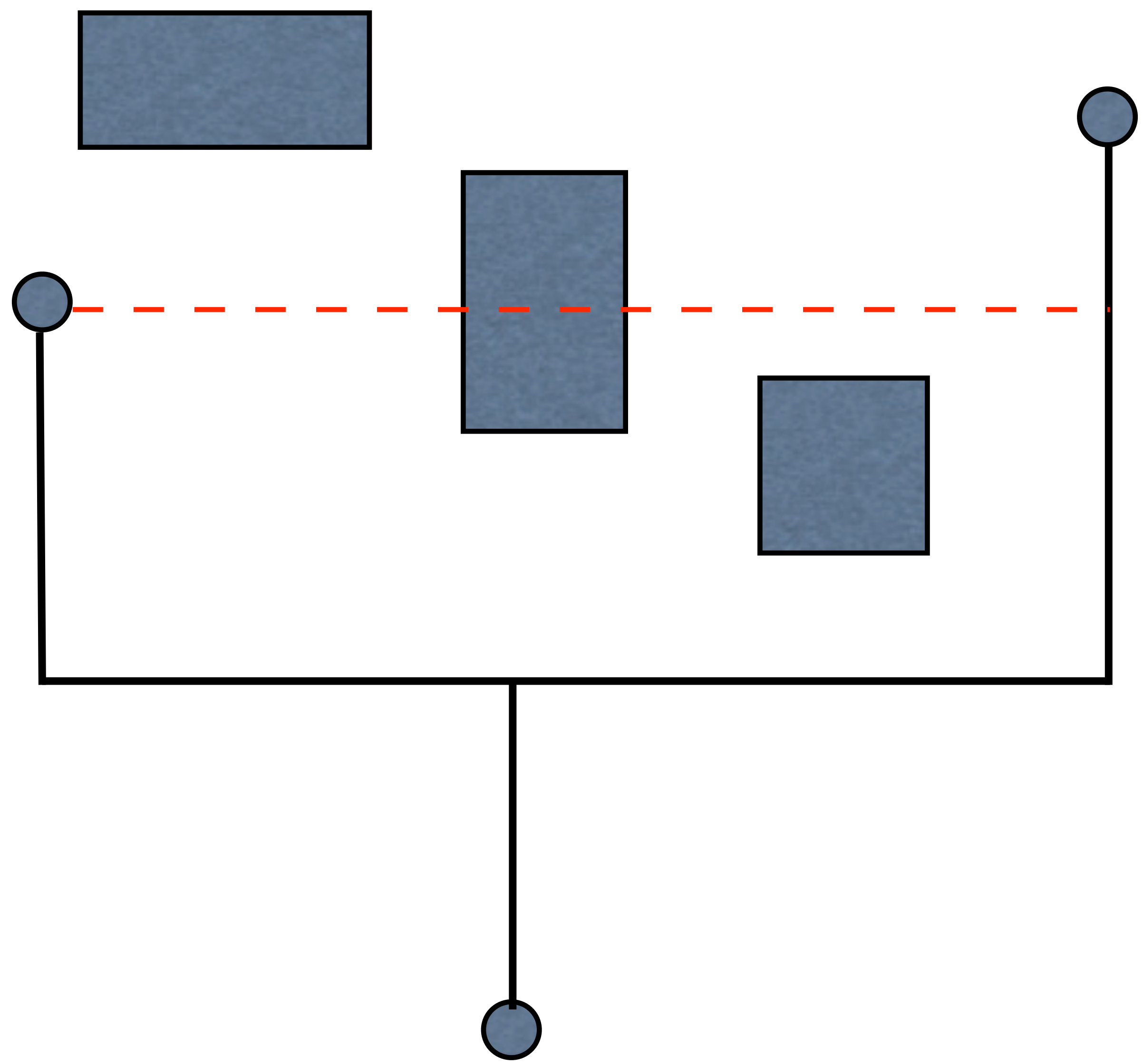


Lower Pocket



Upper Pocket

# Critical Obstacles



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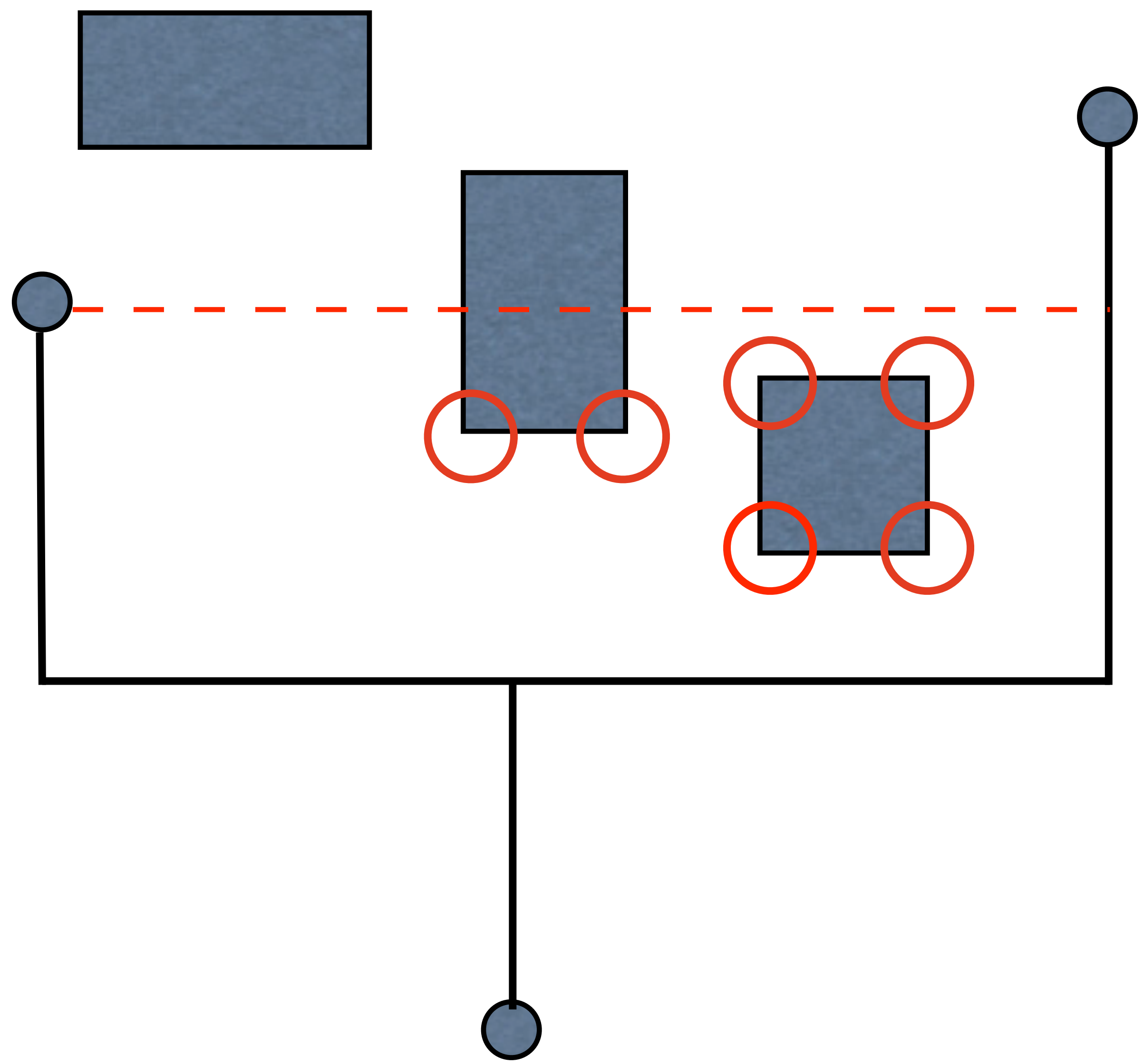

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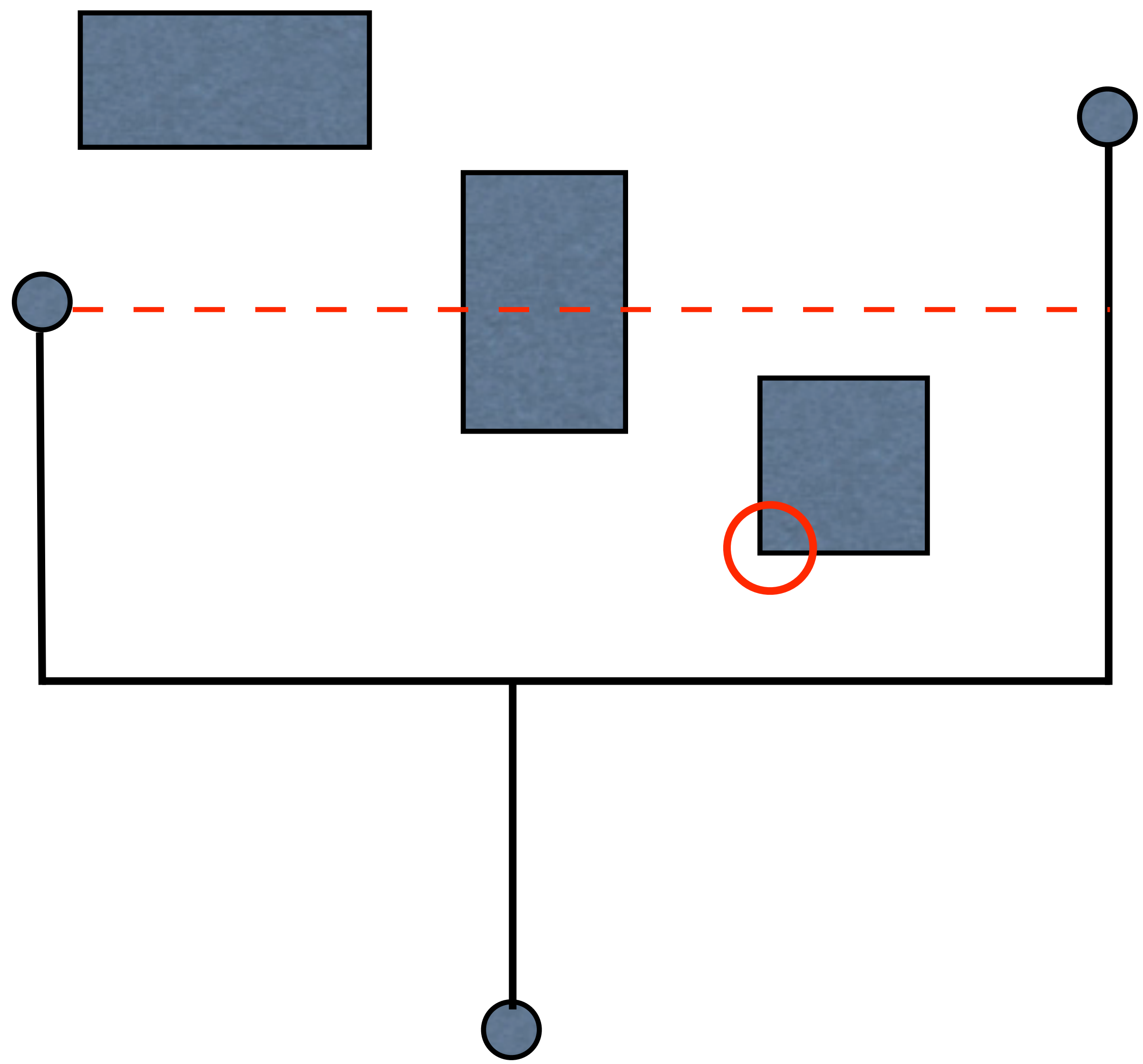
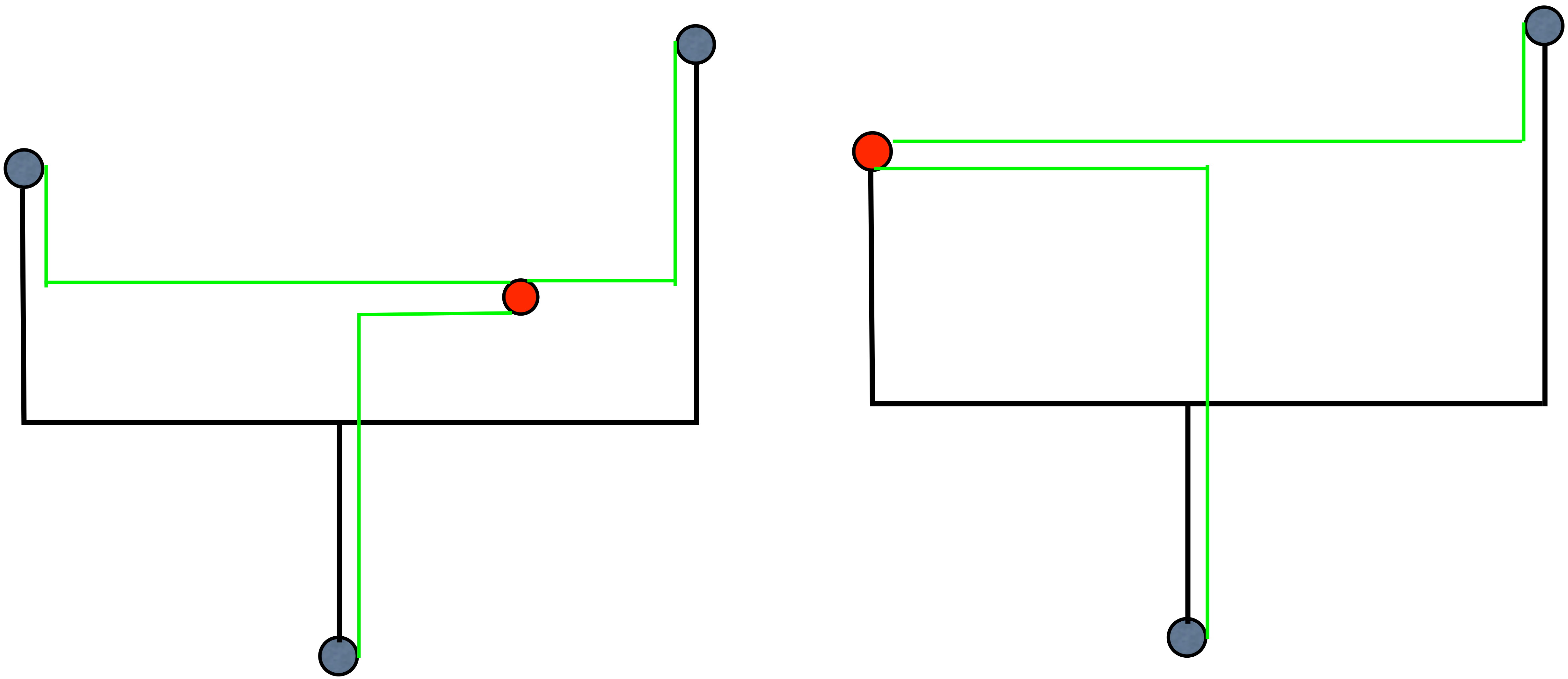
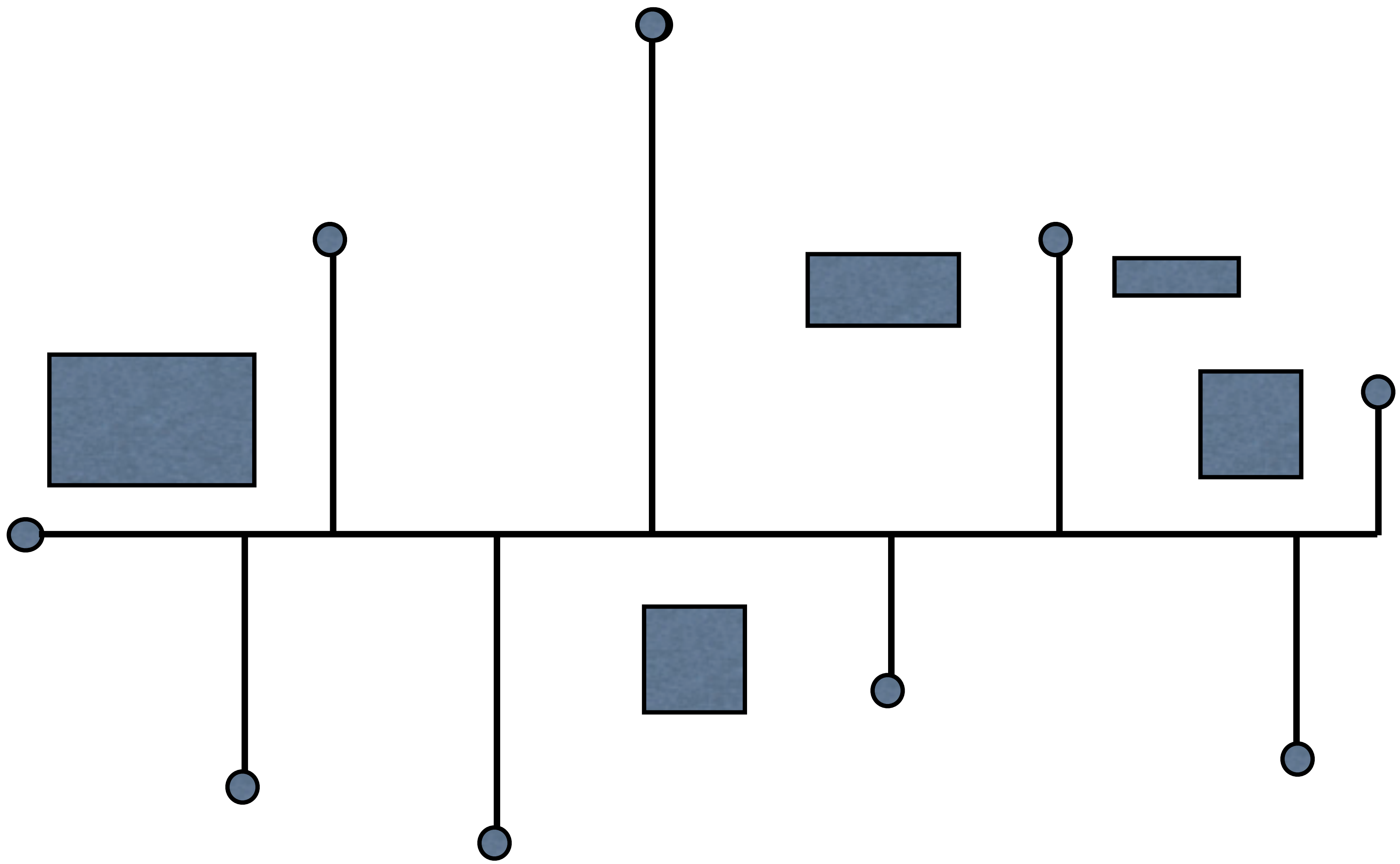


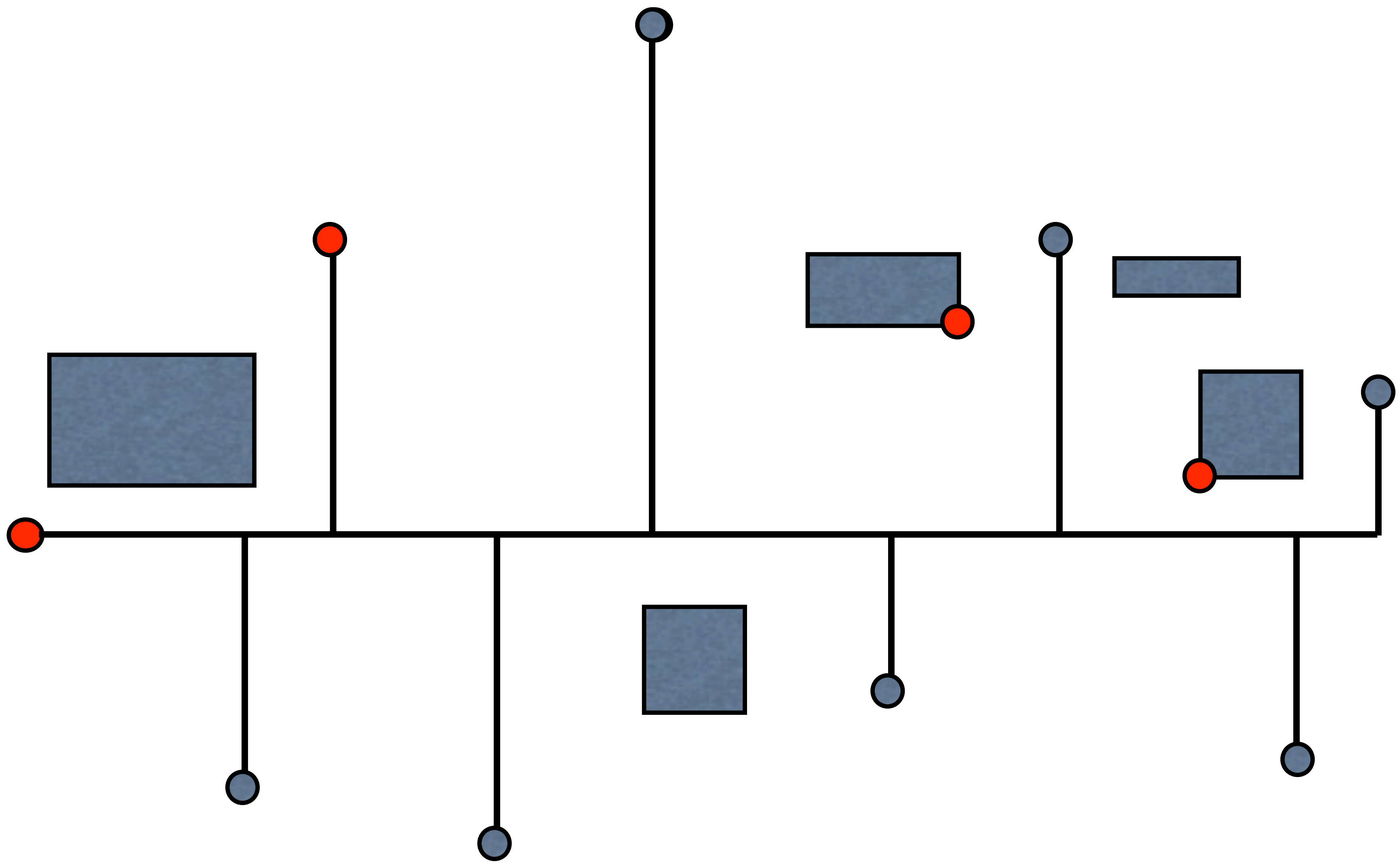
# Green Tree: Upper Pockets



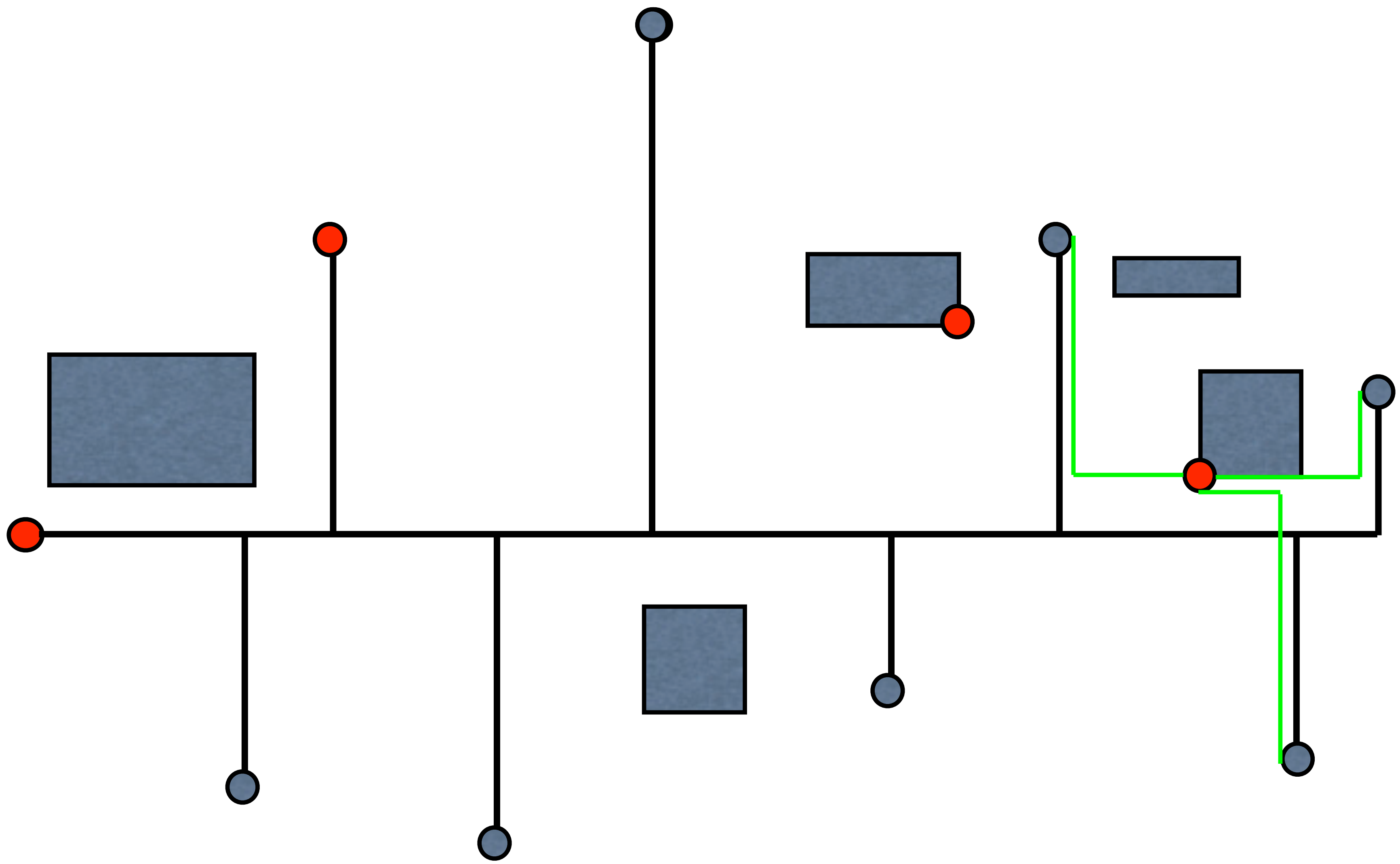
# Green Tree



# Green Tree



# Green Tree



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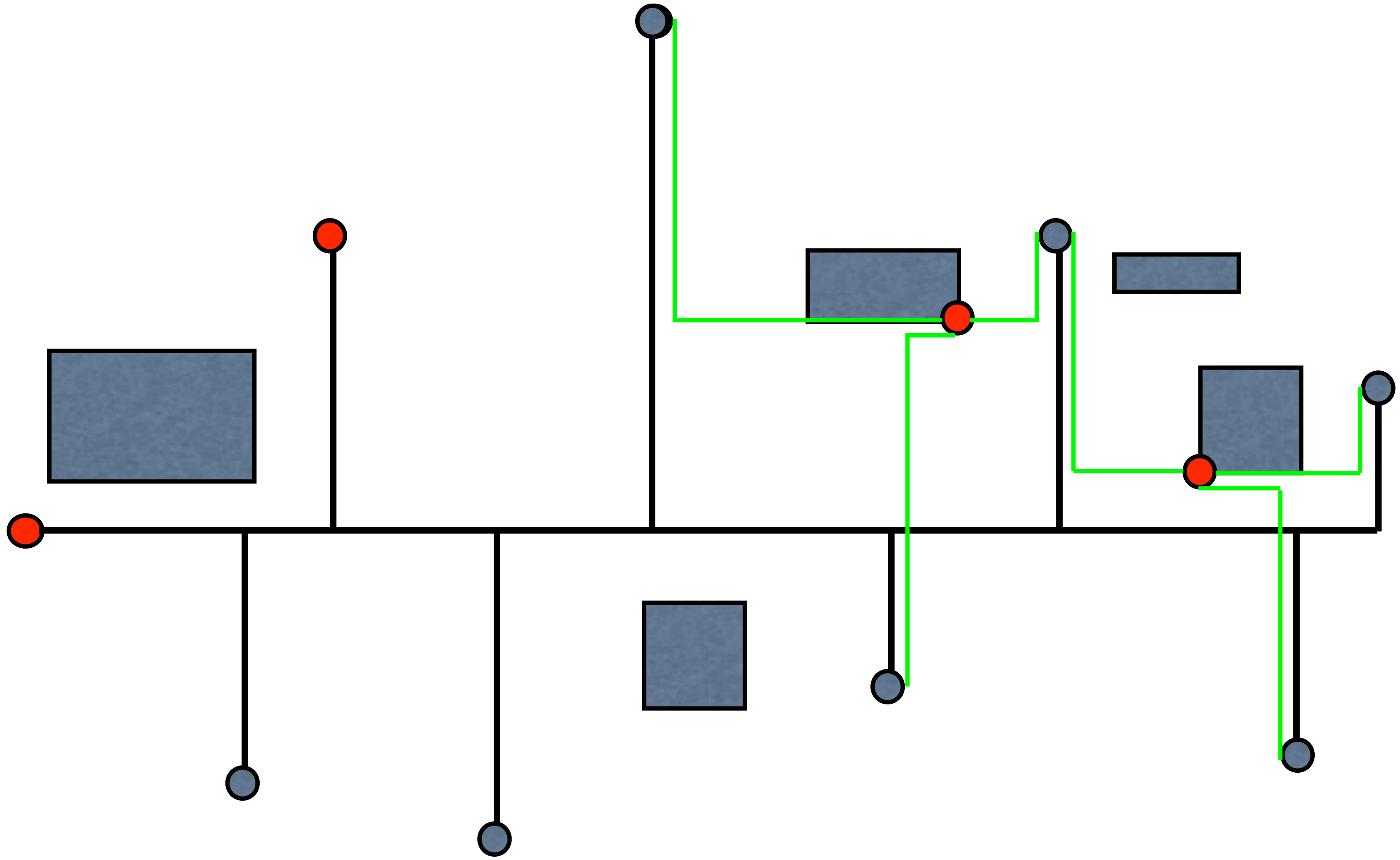

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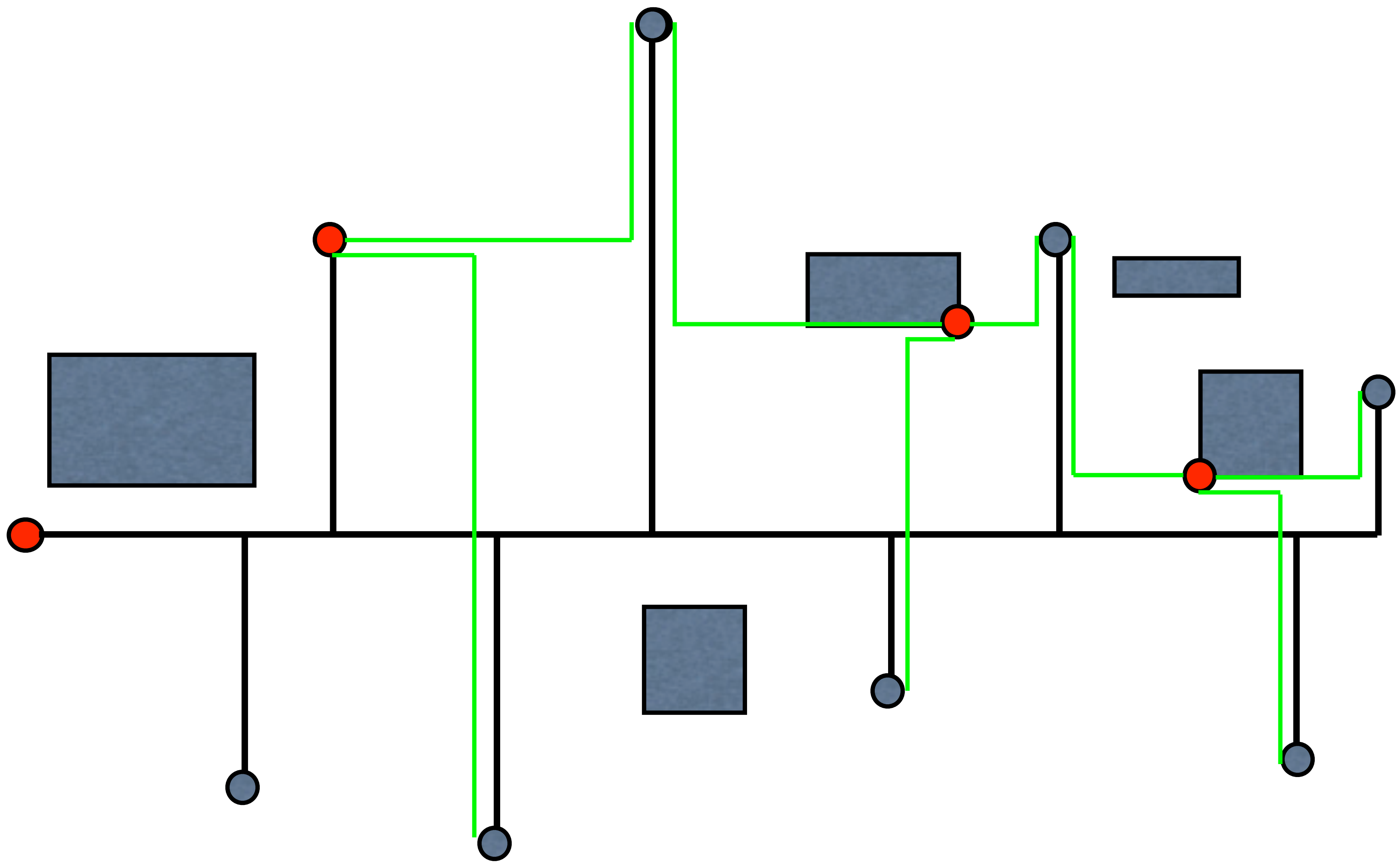

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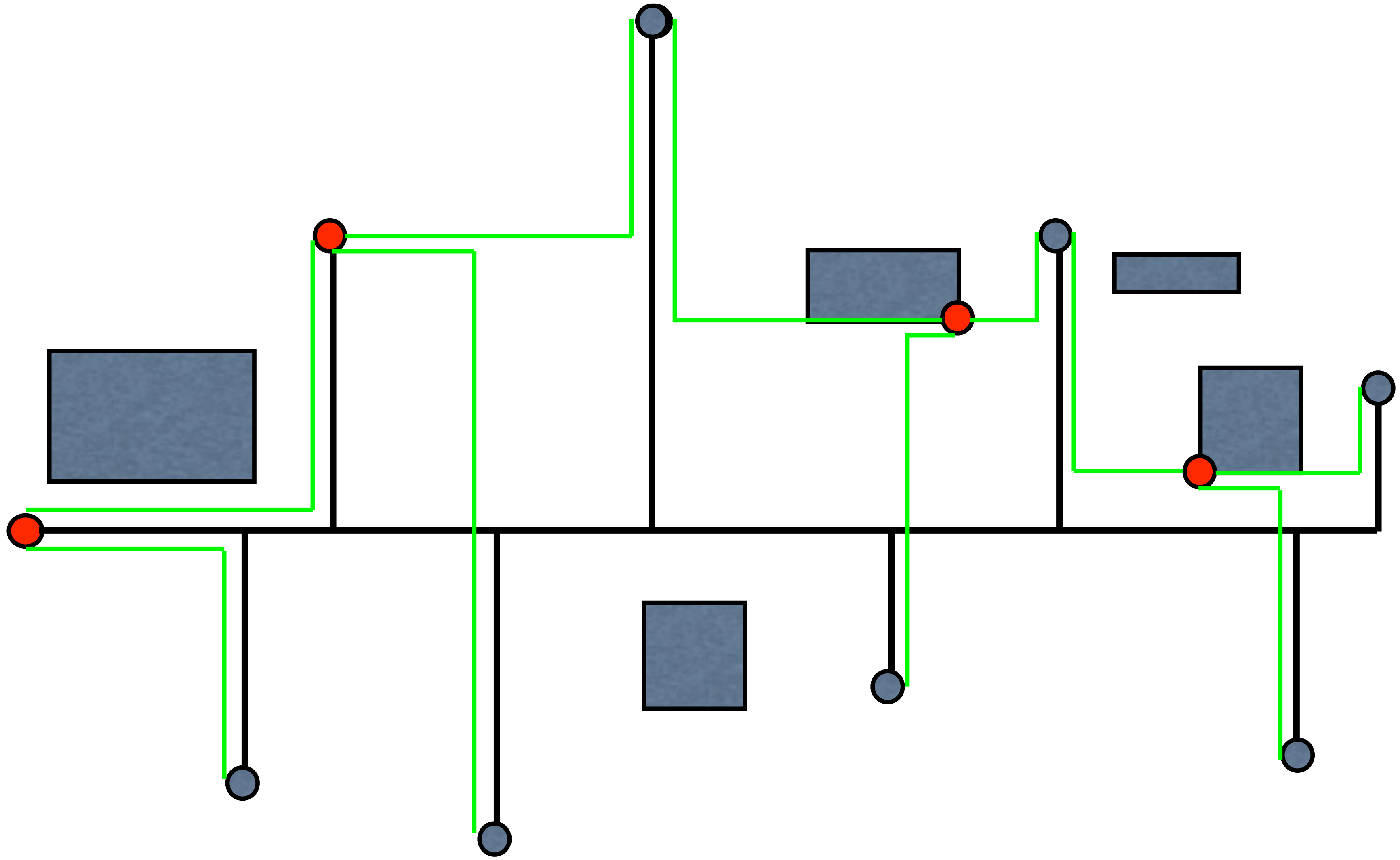

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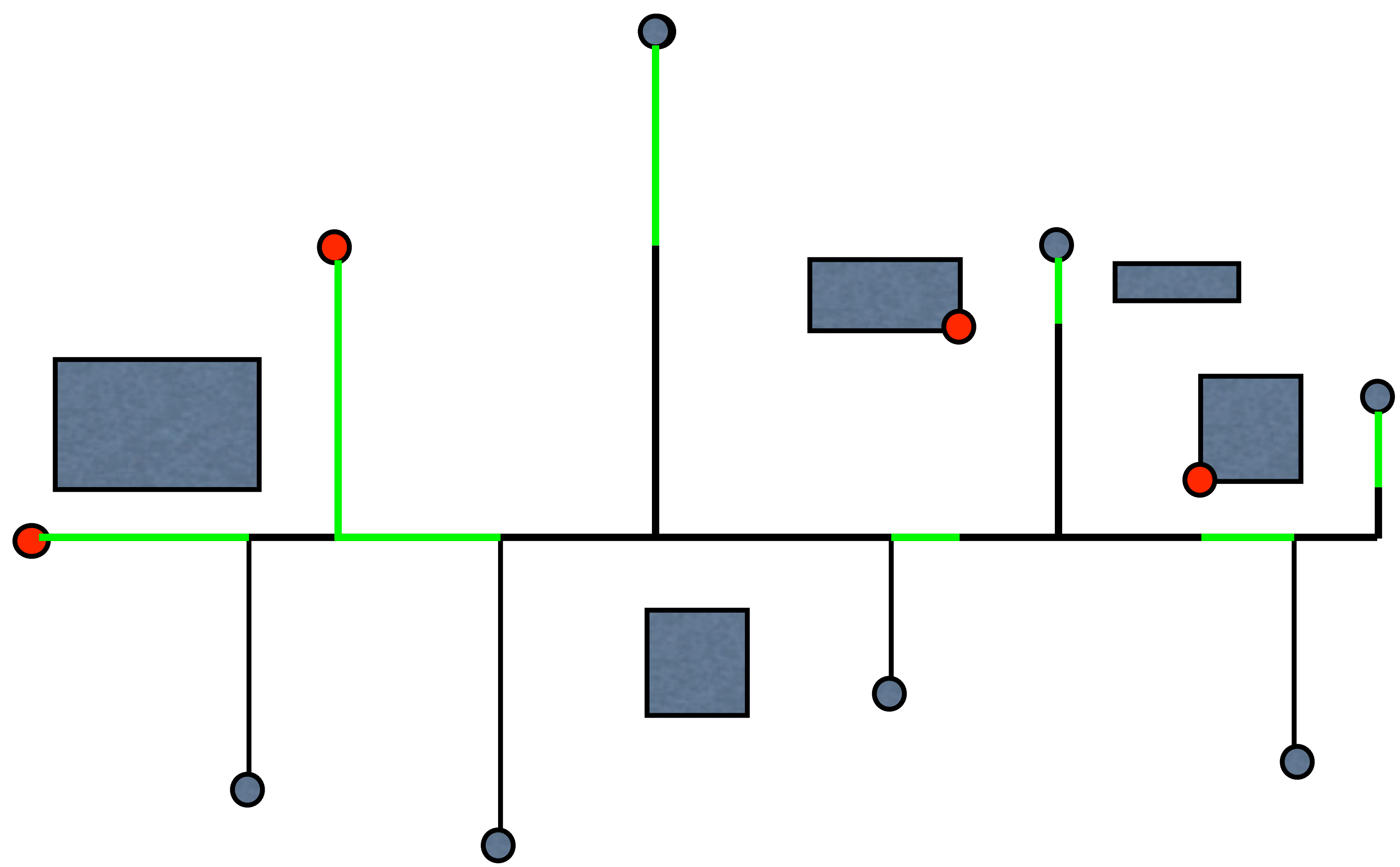
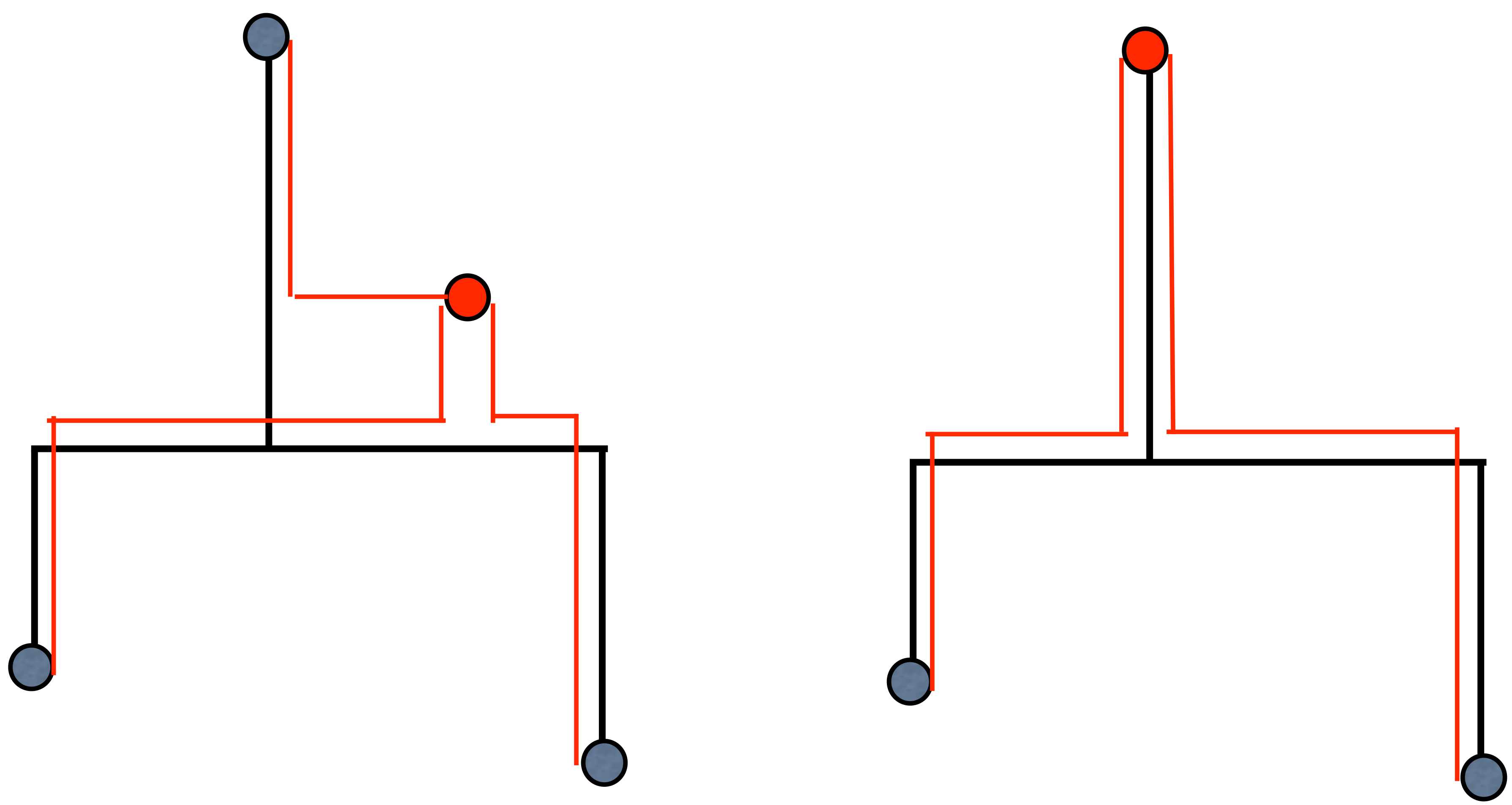
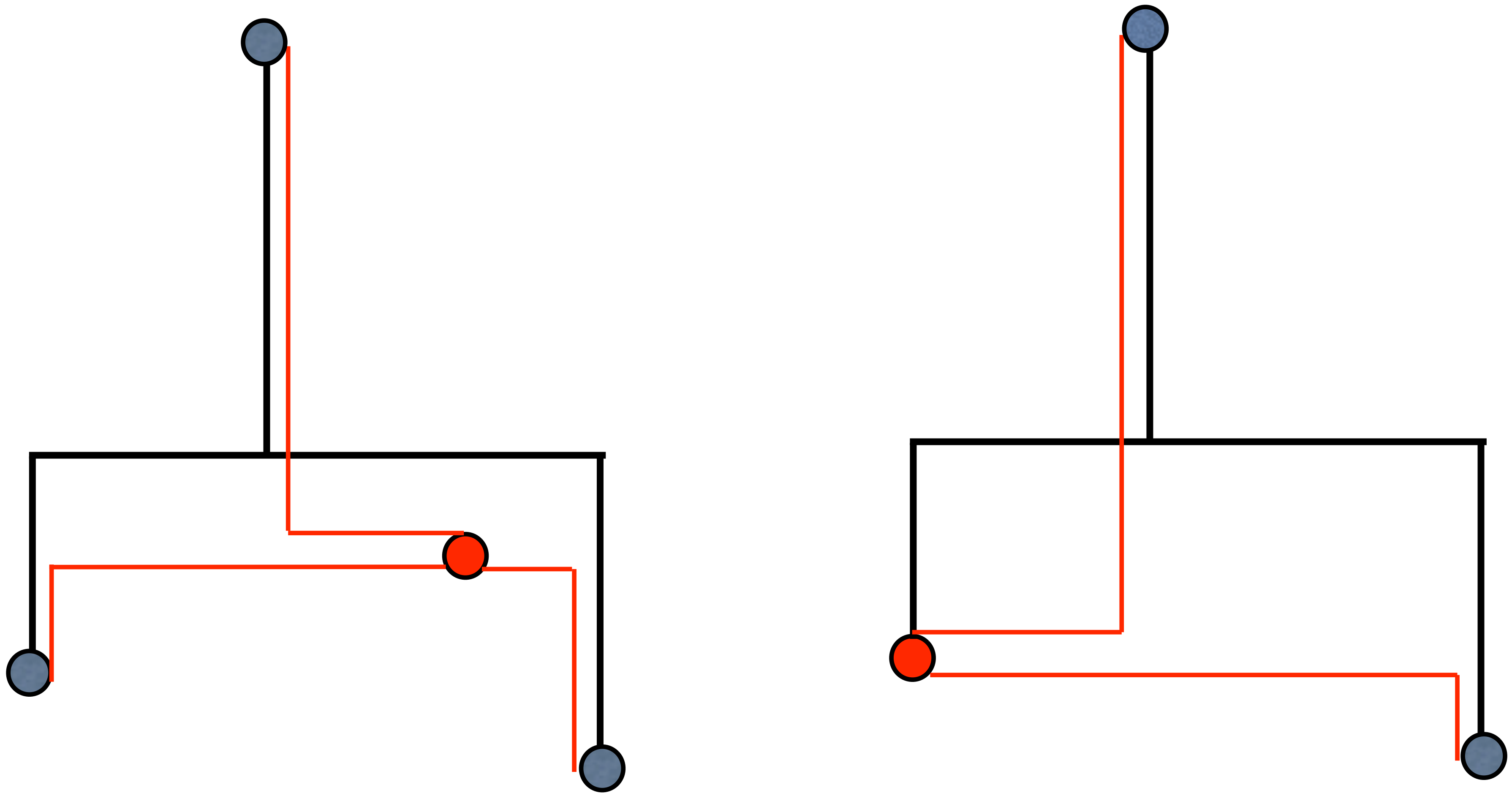


$$|T_{green}| = \sum r_i + 2\sum R_i - \sum H_i^o + |E| + \sum s_i I_i^o$$

# Red Tree: Lower Pockets



# Red Tree: Lower Pockets



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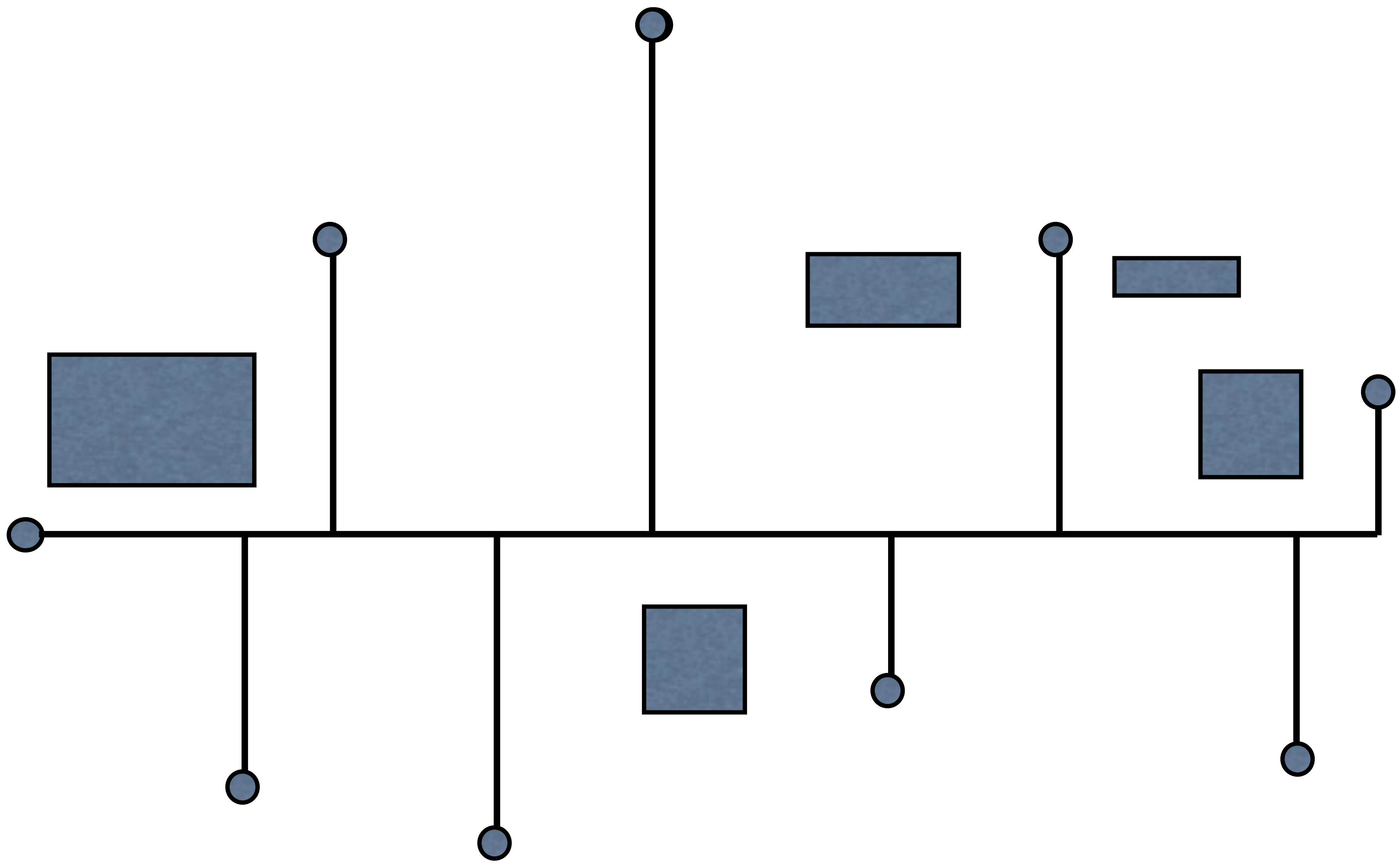

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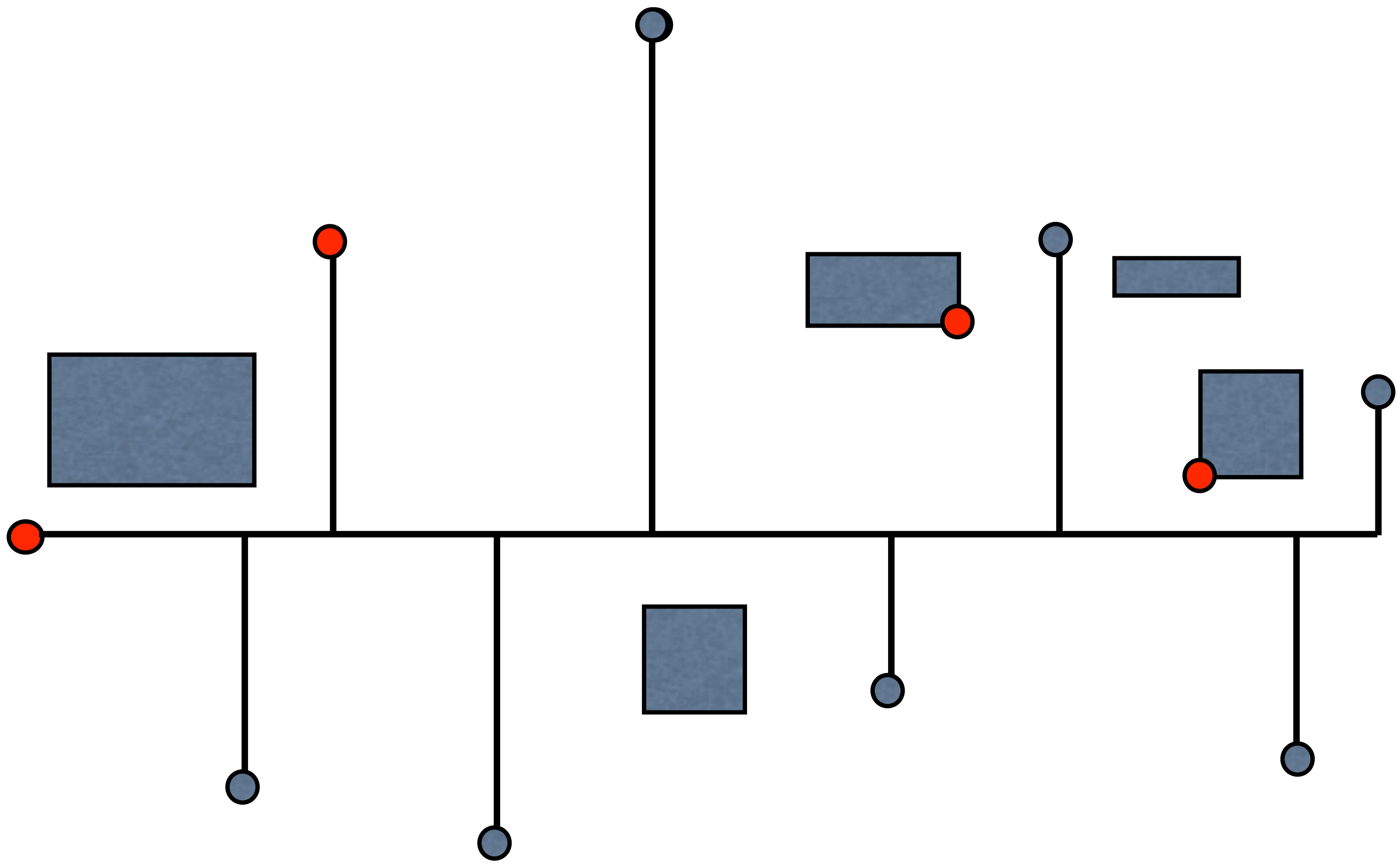

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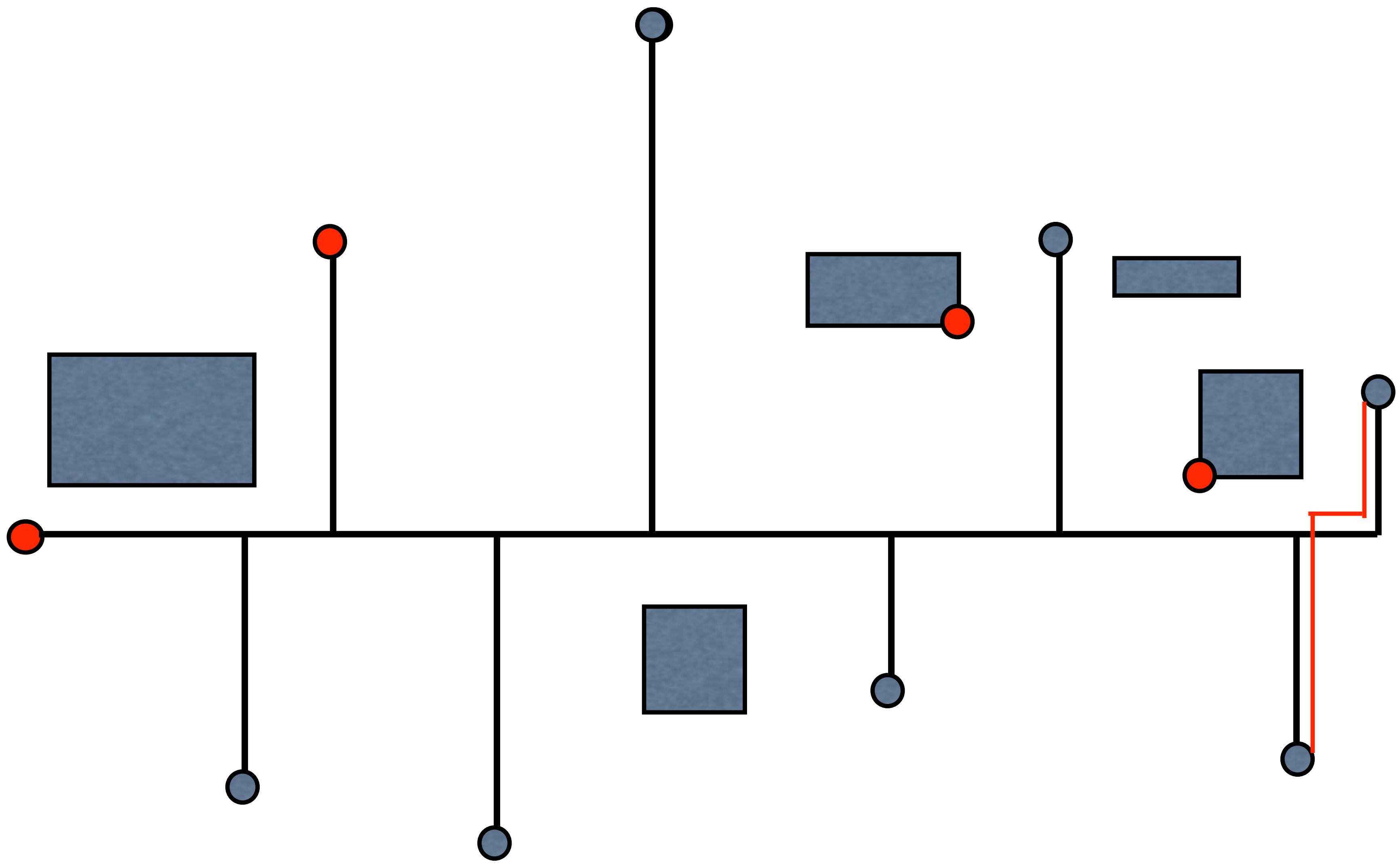

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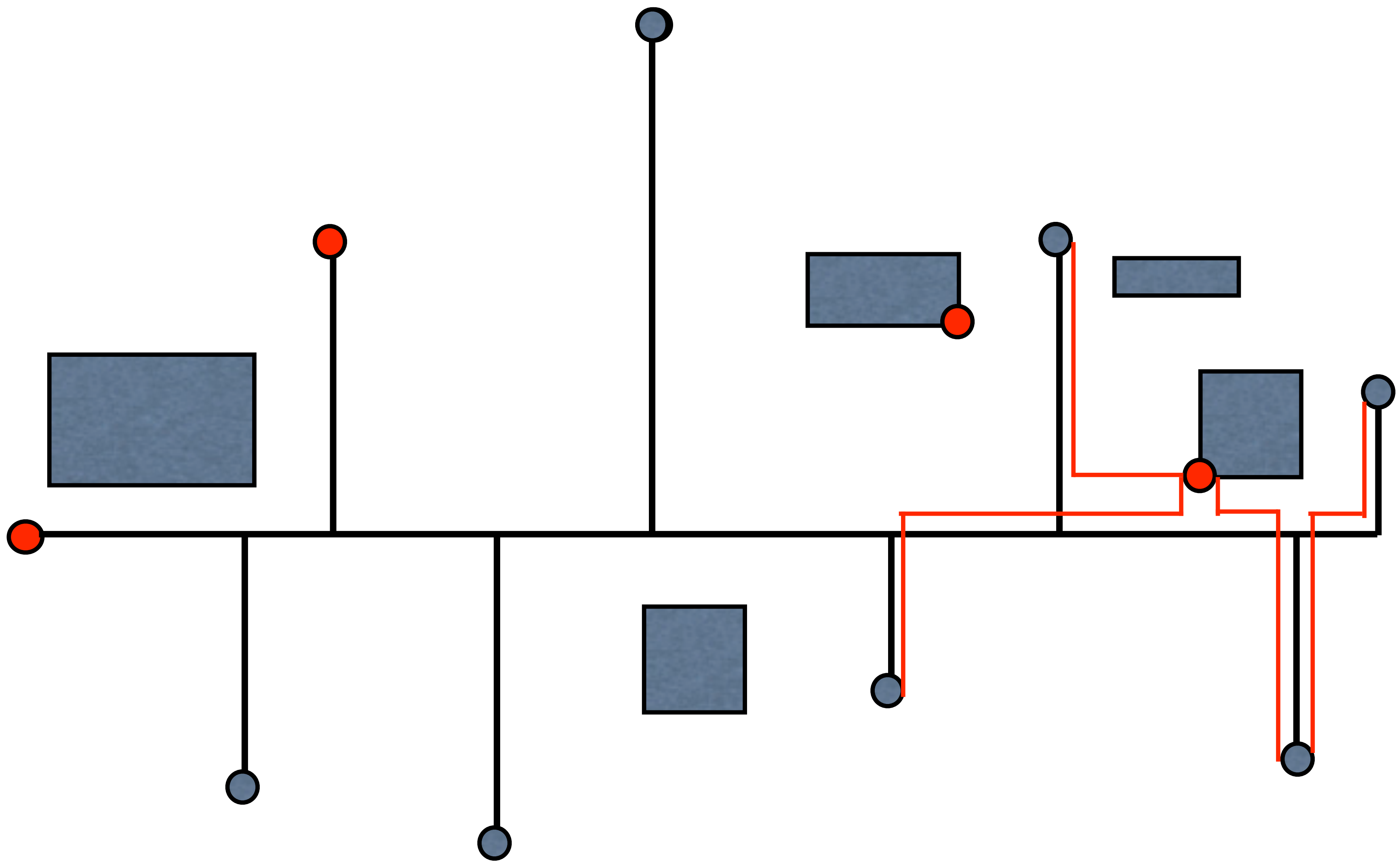

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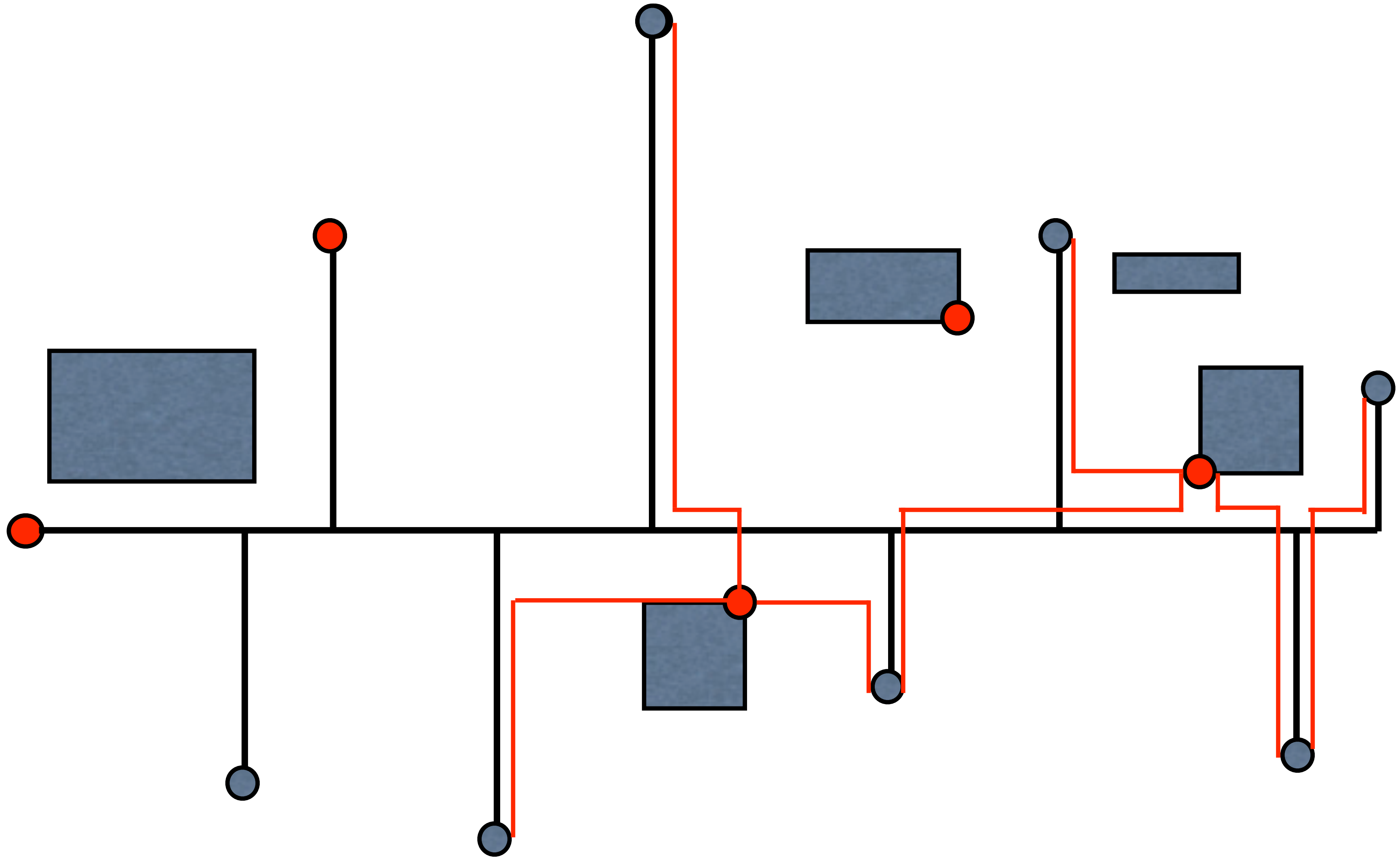

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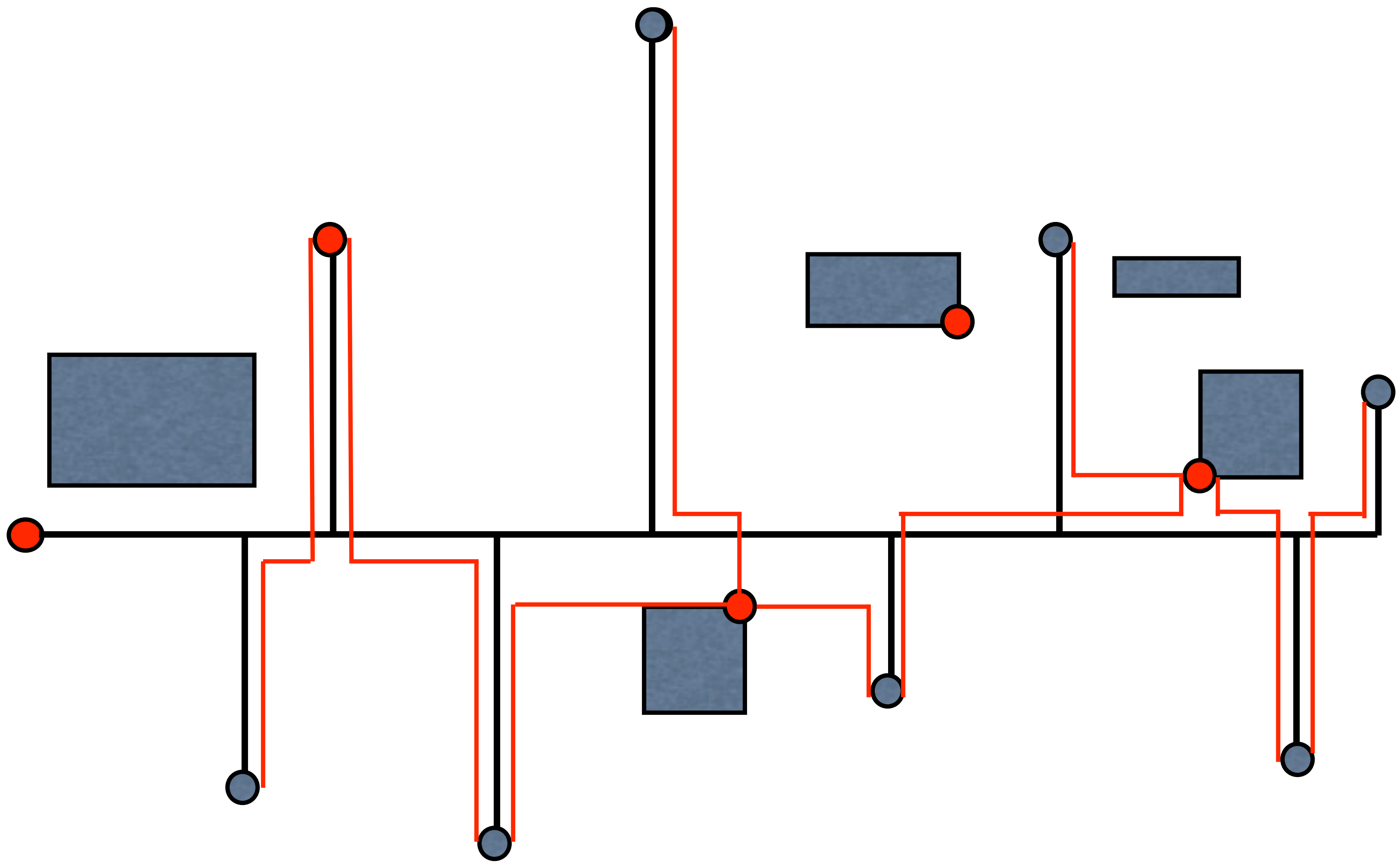

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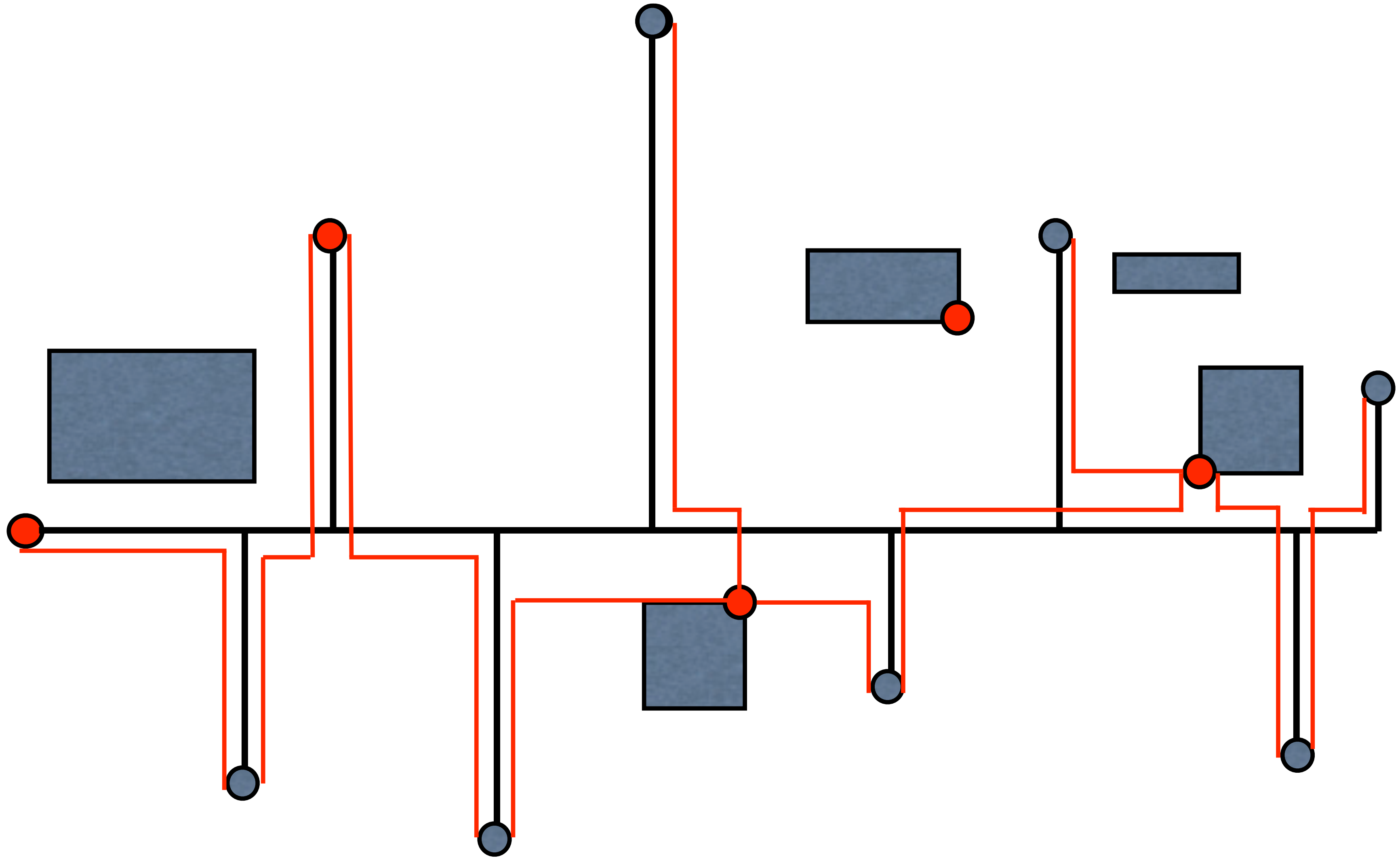

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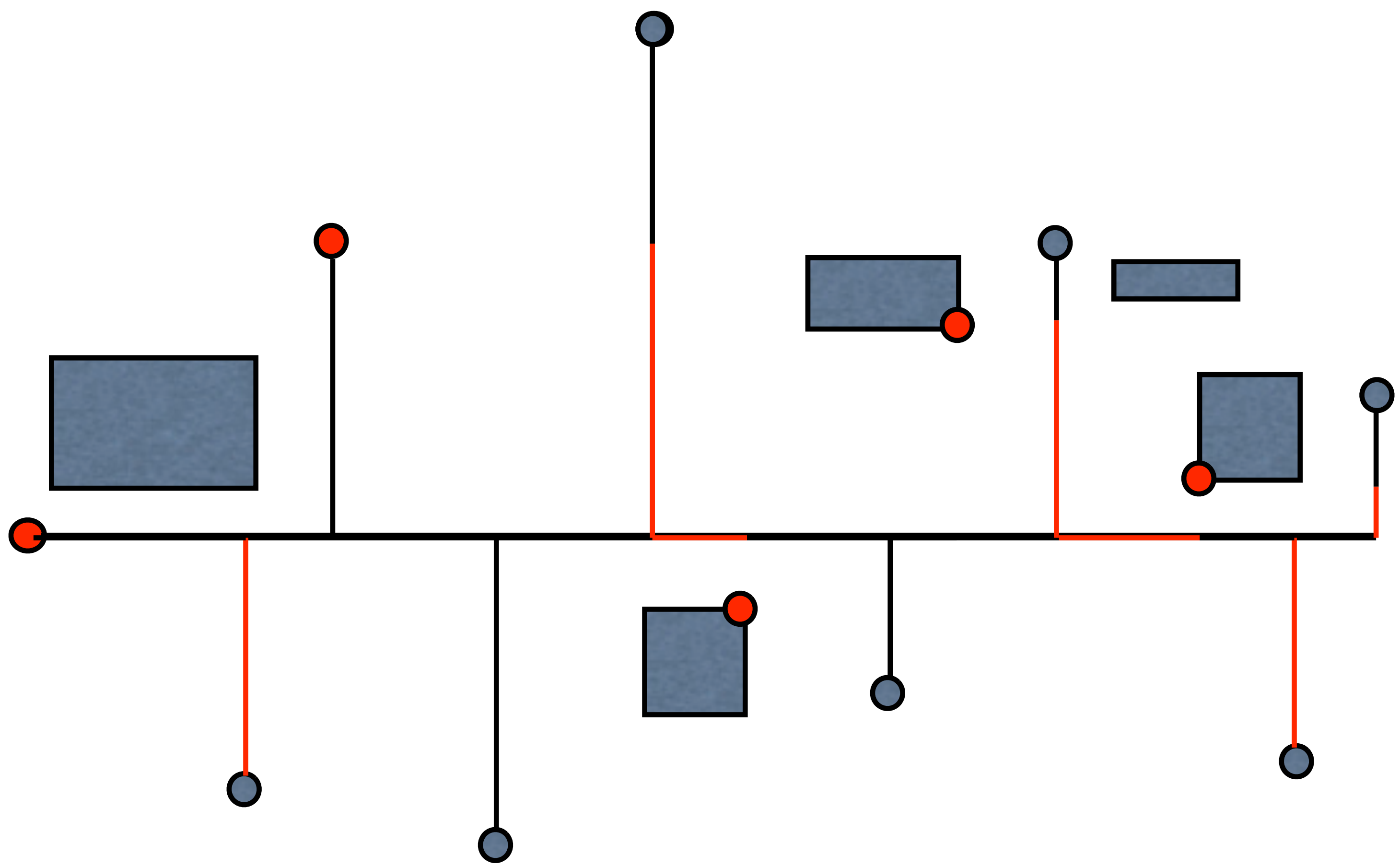
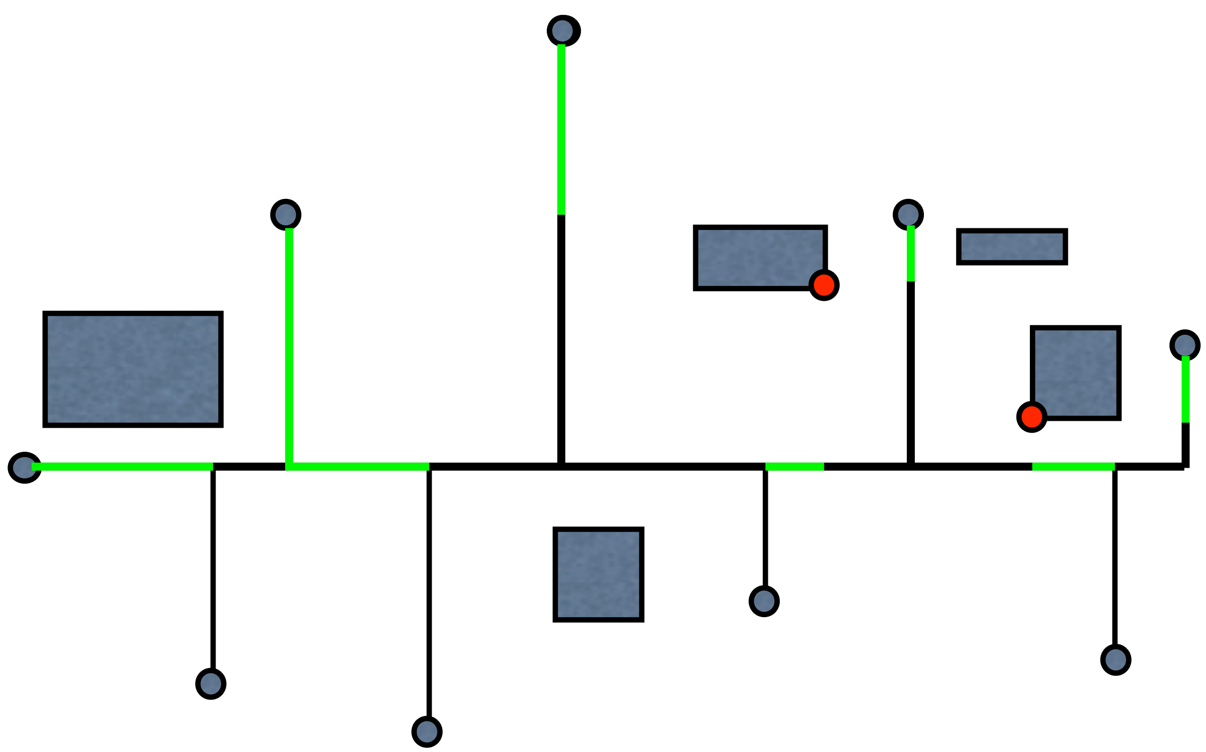
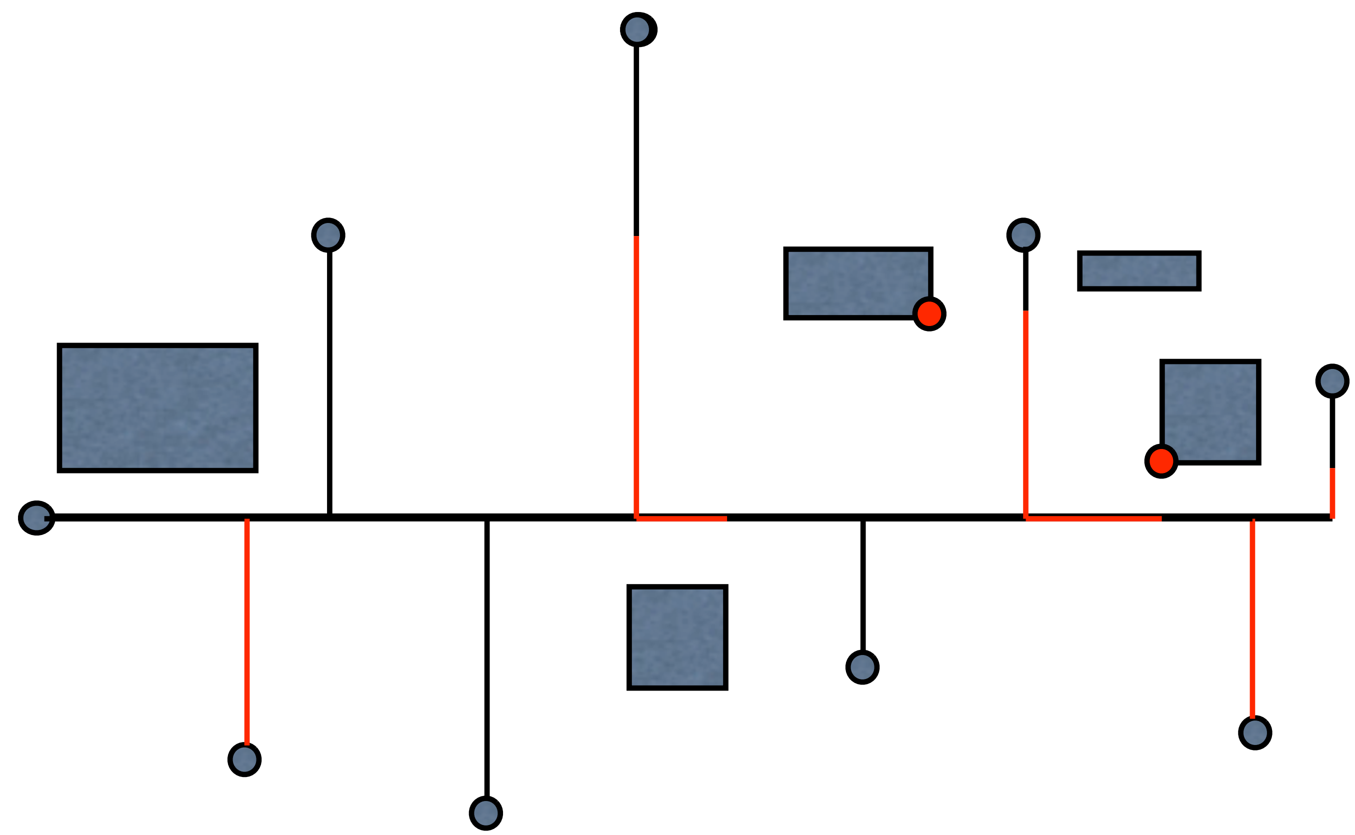


$$|T_{red}| = \sum R_i + 2\sum r_i + \sum H_i^o + 2|E| - \sum s_i I_i^o$$

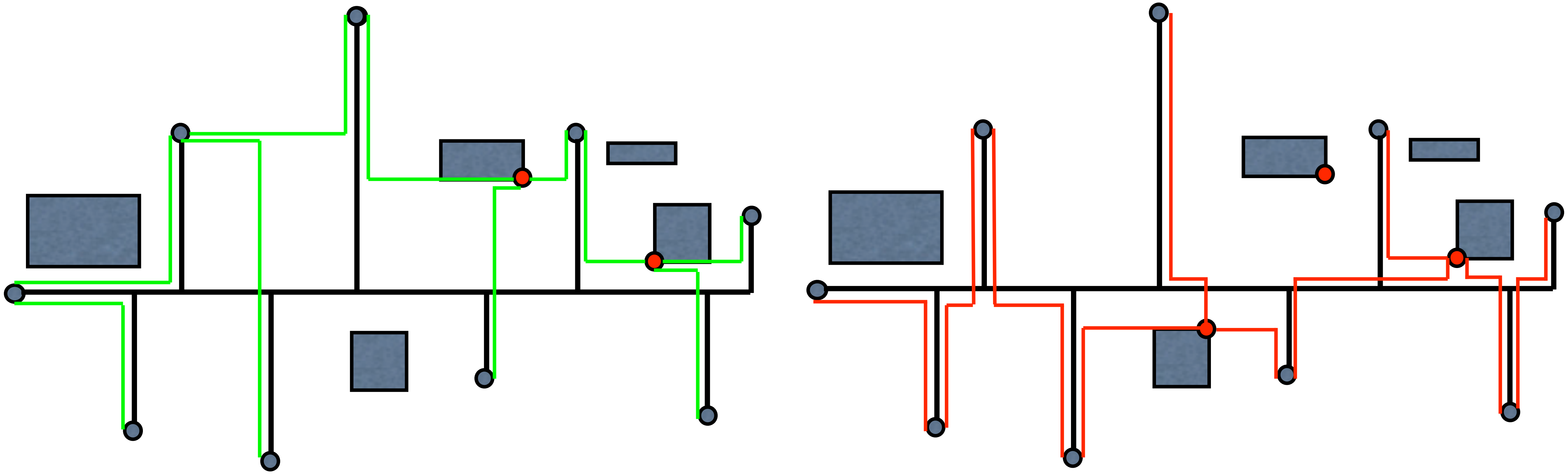
# Red & Green Trees



$$|T_{green}| = \sum r_i + 2\sum R_i - \sum H_i^o + |E| + \sum s_i I_i^o$$



$$|T_{red}| = \sum R_i + 2\sum r_i + \sum H_i^o + 2|E| - \sum s_i I_i^o$$



$$|T_{red}| + |T_{green}| \leq 3|S|$$

$$|AS| \leq \frac{|T_{red}| + |T_{green}|}{2} \leq \frac{3}{2}|S|$$

$\square$

# Algorithm?

- Minimum spanning tree problem is in P.
  - O(nlogn) algorithm
- How do we find a minimum anchored Steiner tree?
- Minimum anchored Steiner tree = Minimum Steiner tree on visibility graph

Hard!

# Approximation Algorithm

- Draw the visibility graph of terminals and obstacles.
- Add vertices at intersections to get a planar graph.
- Apply PTAS for minimum Steiner trees in planar graphs [Borradaile, et al. 2007].
- $(\frac{3}{2} + \epsilon)$-approximation.

# Open Problem

<u>Conjecture:</u>

Euclidean obstacle-avoiding Steiner ratio = $\frac{\sqrt{3}}{2} \simeq 0.866$

Verified for n = 3

Lower bound: $\frac{1}{\sqrt{3}} \simeq 0.577$